CLASSIFYING SPACES OF FINITE ABELIAN p-GROUPS HAVE CHROMATICALLY COMPLETE SUSPENSION SPECTRA

WILLIAM BALDERRAMA

ABSTRACT. Let A be a finite abelian p-group. We record a proof, due to Jeremy Hahn, that the chromatic tower for $\Sigma^\infty BA$ converges.

1. Background

Fix a prime p. For $n \geq 0$, let L_n denote the functor of Bousfield localization with respect to the nth Morava E-theory E_n. These come equipped with maps $L_n \to L_{n-1}$, and the *chromatic tower* of a spectrum X is the resulting tower

$$X \to \cdots \to L_n X \to L_{n-1} X \to \cdots \to L_1 X \to L_0 X.$$

One says that X is *chromatically complete* if the induced map $X_{(p)} \to \lim_{n\to\infty} L_n X$ is an equivalence. The chromatic convergence theorem of Hopkins and Ravenel proves that this holds for all finite spectra:

1.1. **Theorem** ([Rav92, Theorem 7.5.7])**.** Let X be a finite spectrum. Then X is chromatically complete. ◁

In general, it is nontrivial to determine whether a given spectrum is chromatically complete. For example, it is still an open problem to determine whether all suspension spectra are chromatically complete. Perhaps the best general result known is Barthel's generalization of the chromatic convergence theorem:

1.2. **Theorem** ([Bar16])**.** Let X be a connective spectrum of finite projective BP-dimension. Then X is chromatically complete. ◁

Here, the *projective BP-dimension* of a spectrum X is the projective dimension of BP_*X as a BP_*-module. That this indeed generalizes the chromatic convergence theorem follows from a theorem of Adams [Ada69, Lecture 5] that all finite spectra have finite projective BP-dimension.

1.3. **Example.** Johnson and Wilson [JW85] prove that if $A = C_p^n$ is an elementary abelian p-group of rank n, then $\Sigma^\infty_+ BA$ has projective BP-dimension n. In particular, $\Sigma^\infty_+ BA$ is chromatically complete. ◁

Using this, one can prove the following equivariant chromatic convergence theorem:

1.4. **Theorem** ([Bal22, Proposition A.4.11])**.** Let A be an elementary abelian p-group, and write $L_n^b = L_{F(EA_+, i_* E_n)}$, this being a Bousfield localization on the category of A-equivariant spectra. Then

$$X_{(p)} \simeq \lim_{n\to\infty} L_n^b X$$

for any compact A-spectrum X. ◁

Date: August 18, 2023.

At the time, I had thought that the proof would extend to A-equivariant homotopy theory for an arbitrary abelian p-group A provided one could show that classifying spaces of abelian p-groups were chromatically complete. I communicated this at a seminar talk at MIT in February 2023, and later that day Jeremy Hahn sent me a sketch of a very simple and elegant proof of the following:

1.5. **Theorem.** Let A be a finite abelian p-group. Then $\Sigma^\infty_+ BA$ is chromatically complete. ◁

This note records the proof, which is shorter than this introduction. The proof ultimately ends in an application of Theorem 1.2, but not to $\Sigma^\infty_+ BA$. An old conjecture of Landweber says that if A is any p-group of rank n, then $\Sigma^\infty_+ BA$ has projective BP-dimension n, but as far as I know this is wide open except when $n = 1$ or A is elementary abelian. Unfortunately, what I thought was a proof that Theorem 1.5 would imply an A-equivariant chromatic convergence theorem had a gap that I was unable to resolve, but the proof that $\Sigma^\infty_+ BA$ is chromatically complete seems worth recording.

2. The proof

Say that a spectrum X is *BP-projective* if it has projective BP-dimension 0.

2.1. **Lemma.** All spectra in the thick subcategory of $\mathcal{S}\mathrm{p}$ generated by the BP-projective spectra are chromatically complete.

Proof. As $X_{(p)} \to \lim_{n\to\infty} L_n X$ is a natural transformation between exact functors in X, the collection of chromatically complete spectra forms a thick subcategory of $\mathcal{S}\mathrm{p}$. The lemma follows as BP-projective spectra are chromatically complete by Theorem 1.2. □

This might be regarded as a spectral variant of Theorem 1.2, where projective resolutions of BP_*-modules are replaced by BP-projective resolutions of spectra. This allows a great amount of flexibility, as these spectral BP-projective resolutions are not required to be well behaved in BP-homology. With Lemma 2.1 in place, Theorem 1.5 is now an immediate consequence of the following.

2.2. **Proposition.** Let A be an abelian p-group. Then $\Sigma^\infty_+ BA$ is in the thick subcategory of $\mathcal{S}\mathrm{p}$ generated by the BP-projective spectra.

Proof. In general, an elementary abelian p-group A of rank n is isomorphic to a product $C_{p^{t_1}} \times \cdots \times C_{p^{t_n}}$ for some $n \geq 0$ and positive integers $t_1, \ldots, t_n$, in which case

$$\Sigma^\infty_+ BA \simeq \Sigma^\infty_+ BC_{p^{t_1}} \wedge \cdots \wedge \Sigma^\infty_+ BC_{p^{t_n}}.$$

As the collection of BP-projective spectra is closed under smash products, so too is the thick subcategory they generate. We may therefore reduce to $n = 1$, where the claim is that $\Sigma^\infty_+ BC_{p^t}$ is in the thick subcategory generated by the BP-projective spectra.

Write γ for the canonical complex line bundle over $BU(1)$. Then the unit sphere bundle $S(\gamma)$ is the universal $U(1)$-bundle over $BU(1)$, and therefore has contractible total space. It follows that the Cartesian square

$$\begin{array}{ccc} BC_{p^t} & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU(1) & \xrightarrow{p^t} & BU(1) \end{array}$$

of spaces realizes BC_{p^t} as the total space of the unit sphere bundle $S(p^t\gamma)$. As a consequence, if $\mathrm{Th}(p^t\gamma)$ is the Thom spectrum of the complex vector bundle $p^t\gamma$, then there is a cofiber sequence

$$\Sigma^\infty_+ BC_{p^n} \to \Sigma^\infty_+ BU(1) \to \mathrm{Th}(p^t\gamma). \tag{1}$$

Both $\Sigma^\infty_+ BU(1)$ and $\mathrm{Th}(p^t\gamma)$ are BP-projective, the latter by the Thom isomorphism, and this proves the proposition. □

References

[Ada69] J. F. Adams. Lectures on generalised cohomology. Category Theory Homology Theory Appl., Proc. Conf. Seattle Res. Center Battelle Mem. Inst. 1968, Vol. 3, Lect. Notes Math. 99, 1-138 (1969)., 1969.

[Bal22] William Balderrama. Total power operations in spectral sequences. *arXiv e-prints*, page arXiv:2205.07409, May 2022.

[Bar16] Tobias Barthel. Chromatic completion. *Proc. Amer. Math. Soc.*, 144(5):2263–2274, 2016.

[JW85] David Copeland Johnson and W. Stephen Wilson. The Brown-Peterson homology of elementary p-groups. *Amer. J. Math.*, 107(2):427–453, 1985.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.