AN ELEMENTARY PROOF OF $d_2(D_1) = h_0^2 h_3 g_2$

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1. Introduction

In [IX15], Isaksen–Xu produce the Adams differential $d_2(D_1) = h_0^2 h_3 g_2$, completing the identification of the stable 51 and 52-stems. Their proof uses motivic homotopy theory in an essential way, and is notable for the fact that no non-motivic proof was known at the time. A non-motivic and completely automated proof should now follow from Chua's algorithmic computation of Adams d_2-differentials [Chu21], building on Baues–Jibladze's work on the secondary Steenrod algebra [BJ04].

This note gives another proof by analyzing how these elements arise in the unstable Adams spectral sequence. The proof is elementary, in the sense that all the methods were available in the 70s and the necessary Ext computations were available in the 90s.

2. Preliminaries

Write $E(S^n)$ for the unstable Adams spectral sequence for S^n. We name classes in the unstable Adams spectral sequence by how they appear in a Curtis table. See [Cur71, CGMM87] for background.

2.1. **Lemma.** In a Curtis table, we have

$$D_1 = \lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}, \qquad h_0^2 h_3 g_2 = \lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7.$$

Proof. That $D_1 = \lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}$ is given in [Tan94]. That $h_0^2 h_3 g_2 = \lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7$ may be read off [CGMM87], as this is the only nonzero class in its degree. □

Note especially that these classes have the same initial. Consider the map

$$H\colon E_1(S^5) \to E_1(S^9), \qquad H(\lambda_u\lambda_I) = \begin{cases} \lambda_I & u = 4 \\ 0 & \text{otherwise} \end{cases}$$

of spectral sequences, converging to the Hopf map $H\colon \Omega S^5 \to \Omega S^9$.

2.2. **Lemma.** In a Curtis table, we have

$$\lambda_7\lambda_{11}\lambda_{15}\lambda_{15} = h_3 c_2, \qquad \lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7 = h_0 h_2 g_2.$$

Proof. These are the only nonzero classes in their respective degrees, so the lemma may be read off [CGMM87]. □

2.3. **Lemma.** There is an Adams differential $d_2(h_3 c_2) = h_0 h_2 g_2$.

Proof. Barratt–Mahowald–Tangora [BMT70] produce an Adams differential $d_2(c_2) = h_0 f_1$. It follows that $d_2(h_3 c_2) = h_0 h_3 f_1$, so it suffices to show $h_3 f_1 = h_2 g_2$. This is obtained by applying Sq^0 to the relation $h_2 f_0 = h_1 g$ in the 21-stem. □

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3. The proof

3.1. **Theorem.** There is an Adams differential $d_2(D_1) = h_0^2 h_3 g_2$.

Proof. The classes D_1 and $h_0^2 h_3 g_2$ desuspend to $\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}$ and $\lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7$ in $E_2(S^5)$ by Lemma 2.1. Moreover, we may read off [CGMM87] that $\lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7$ is the only nonzero class in its degree in $E_2(S^5)$. It follows that

$$d_2(\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}) = \alpha \cdot \lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7 \tag{1}$$

in $E_2(S^5)$ for some $\alpha \in \mathbb{F}_2$. We then have

$$\begin{aligned} d_2(\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}) &= d_2(H(\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15})) = H(d_2(\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15})) \\ &= H(\alpha \cdot \lambda_4\lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7) = \alpha \cdot \lambda_6\lambda_9\lambda_{11}\lambda_7\lambda_7\lambda_7 \end{aligned}$$

in $E_2(S^9)$. By Lemma 2.2, this suspends to

$$d_2(h_3 c_2) = \alpha \cdot h_0 h_2 g_2$$

in the stable Adams spectral sequence, and so $\alpha = 1$ by Lemma 2.3. Thus Eq. (1) suspends to $d_2(D_1) = h_0^2 h_3 g_2$ as claimed. □

References

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