THE EQUIVARIANT $K(n)$-LOCAL SPHERE

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Fix a finite group G, and write $\nu_G\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}^G$ for the functor sending a spectrum X to the Borel G-spectrum on X with trivial action.

0.0.1. **Theorem.** There is an equivalence $L_{\nu_G(K(n))}S_G \simeq \nu_G(S_{K(n)})$. ◁

To prove Theorem 0.0.1, we must verify that $\nu_G(S_{K(n)})$ is $\nu_G(K(n))$-local, and that the unit map $S_G \to \nu_G(S_{K(n)})$ is an $\nu_G(K(n))$-equivalence.

0.0.2. **Proposition.** $\nu_G(S_{K(n)})$ is $\nu_G(K(n))$-local.

Proof. Fix X which is $\nu_G(K(n))$-acyclic; we must show that $[X, \nu_G(S_{K(n)})] = 0$. Note that

$$[X, \nu_G(S_{K(n)})] \cong [X_{\mathrm{h}G}, S_{K(n)}].$$

Thus it is sufficient to verify that $X_{\mathrm{h}G}$ is $K(n)$-acyclic. Indeed, as $\nu_G(K(n))$ is a ring, X is $\nu_G(K(n))$-cohomologically acyclic. As

$$[X, \nu_G(K(n))] \cong [X_{\mathrm{h}G}, K(n)],$$

it follows that $X_{\mathrm{h}G}$ is $K(n)$-cohomologically acyclic. As $K(n)$ is a field, this implies that $X_{\mathrm{h}G}$ is $K(n)$-acyclic as desired. □

0.0.3. **Lemma.** Let R be a Borel G-equivariant ring spectrum. Then $R^{tG} = 0$ if and only if the H-geometric fixed points $\Phi^H R$ vanish for all nontrivial subgroups $H \subset G$.

Proof. Combine [MNN15a, Proposition 2.13] and [MNN15b, Theorem 6.41]. □

0.0.4. **Lemma.** $\Phi^H \nu_G(K(n))$ vanishes for all nontrivial subgroups $H \subset G$.

Proof. Combine Lemma 0.0.3 and [GS96, Theorem 1.1]. □

0.0.5. **Proposition.** $S_G \to \nu_G(S_{K(n)})$ is a $\nu_G(K(n))$-equivalence.

Proof. After smashing with $\nu_G(K(n))$, this map becomes

$$\nu_G(K(n)) \to \nu_G(K(n)) \otimes \nu_G(S_{K(n)}).$$

To show this is an equivalence, it suffices to verify that the map

$$\Phi^H \nu_G(K(n)) \to \Phi^H(\nu_G(K(n)) \otimes \nu_G(S_{K(n)})) \simeq \Phi^H \nu_G(K(n)) \otimes \Phi^H \nu_G(S_{K(n)})$$

is an equivalence for all subgroups $H \subset G$. If $H = e$, then this is the map $K(n) \to K(n) \otimes S_{K(n)}$, which is an equivalence by definition. If H is nontrivial, then both sides vanish by Lemma 0.0.4. □

Combining Proposition 0.0.2 and Proposition 0.0.5 proves Theorem 0.0.1.

0.0.6. **Remark.** If G is a p-group, then $\nu_G(K(1)) \simeq KU_G/(p)$ [Ati61, Theorem 7.2] [AT69, §1, Proposition 1.1], and thus $\nu_G(S_{K(1)}) \simeq L_{KU_G/(p)}S_G$. ◁

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References

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