# THE ALGEBRAIC TRANSFINITE (TRANSFINITE ALGEBRAIC?) ATIYAH–HIRZEBRUCH SPECTRAL SEQUENCE

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Abstract. We describe the Curtis algorithm for computing the $E_2$-page of the Adams spectral sequences for the spectra $L(k)_n$. This note is companion to the file goodwillie-curtis-table.txt, which contains the output of this algorithm through stem 48.

0.1. **Sequences.** Say that a sequence $J = (j_1, \ldots, j_k)$ is CU if $j_s > 2j_{s+1}$ and $j_s \geq 1$ for each $s$. Write $|J| = k$ for the length of such a sequence, $e(J) = j_k$ for its last term, and $||J|| = j_1 + \cdots + j_k$ for the sum of its terms. Define an ordering on CU sequences by declaring $J < J'$ if $j_k < j'_k$, or else $j_k = j'_k$ and $j_{k-1} < j'_{k-1}$, and so forth. Say that a sequence $I = (i_1, \ldots, i_l)$ is admissible if $2i_t \geq i_{t+1}$ and $i_t \geq 0$ for each $t$. Define an ordering on admissible sequences by $I < I'$ if $i_1 < i'_1$, or else $i_1 = i'_1$ and $i_2 < i'_2$, and so forth.

0.2. **The TAHSS.** We work at the prime 2. The *Goodwillie spectral sequence* (GSS) for $S^n$ is a spectral sequence of signature

$$E_1 = \bigoplus_{k \geq 0} \pi_t L(k)_n \Rightarrow \pi^u_{t+n-k}(S^n),$$

where $L(k)_n$ are certain spectra. Here, we write $\pi_*$ for stable homotopy groups, and $\pi^u_*$ for unstable homotopy groups. To understand the GSS, one must first understand something about $L(k)_n$. In [Beh12], Behrens develops a method for computing $\pi_* L(k)_n$, the *transfinite Atiyah–Hirzebruch spectral sequence* (TAHSS), which goes as follows.

There are suspension maps $L(k)_n \to L(k)_{m+1}$, and we define

$$L(k)^m_n = \mathrm{Fib}\left(L(k)_n \to L(k)_{m+1}\right).$$

There is then a filtration

$$\cdots \to L(k)^m_n \to L(k)^{m+1}_n \to L(k)_n,$$

with filtration quotients described by cofiber sequences

$$L(k)^{m-1}_n \longrightarrow L(k)^m_n \longrightarrow \Sigma^m L(k-1)_{2m+1}. \tag{1}$$

The spectrum $L(k-1)_{2m+1}$ itself has a filtration with filtration quotients built from $L(k-2)_*$, and so forth. As $L(0)_t = S$, altogether these give rise to a transfinite spectral sequence

$$\bigoplus_{\substack{j_1, \ldots, j_k \\ j_s > 2j_{s+1},\, j_k \geq n}} \pi_t S^{j_1 + \cdots + j_k} \Rightarrow \pi_t L(k)_n,$$

*Date*: August 16, 2022.

which we call the TAHSS for $L(k)_n$. Given a sequence $J$, write $[J]$ for the generator of the summand corresponding to $J$. We can then rewrite the TAHSS for $L(k)_n$ as

$$\pi_* S\{[J] : J \text{ CU}, e(J) \geq n\} \Rightarrow \bigoplus_{k\geq 0} \pi_* L(k)_n, \qquad |[J]| = ||J||.$$

We point out two benefits of this approach. First, the TAHSS for $L(k)_n$ is determined by the TAHSS for $L(k) = L(k)_1$, allowing for a computation which is uniform in $n$. Second, differentials in the TAHSS for $L(k)$ are in bijective correspondence with a specific subset of the differentials in the TAHSS for $L(k+1)$, allowing for a computation which is inductive in $k$.

0.3. **The ATAHSS.** There is a third benefit. Differentials in the TAHSS for $L(k)$ are governed by the structure of attaching maps in the given cell structure of $L(k)$. A great number of these are detected in homology. This is one of the main techniques of [Beh12]. Let

$$\mathcal{L}(k)_n = H_* L(k)_n, \qquad \mathcal{L}(k)_n^m = H_* L(k)_n^m.$$

As before, when $n = 1$ we omit it from the notation. We then have

$$\mathcal{L}(k)_n^m = \mathbb{F}_2\{\bar{Q}^J : J \text{ CU}, |J| = k, n \leq e(J) \leq m\}, \qquad |\bar{Q}^J| = ||J||,$$ [1]

and the cofiber sequences of Eq. (1) induce short exact sequences on homology, with maps given by

$$\mathcal{L}(k)_n^{m-1} \to \mathcal{L}(k)_n^m, \qquad \bar{Q}^J \mapsto \bar{Q}^J$$

and

$$\mathcal{L}(k)_n^m \to \mathcal{L}(k-1)_{2m+1}, \qquad \bar{Q}^{j_1} \cdots \bar{Q}^{j_k} \mapsto \begin{cases} \bar{Q}^{j_1} \cdots \bar{Q}^{j_{k-1}} & j_k = m; \\ 0 & j_k \neq m. \end{cases}$$

Let $\mathcal{A}$ denote the dual Steenrod algebra, and write

$$\mathrm{Ext}(k)_n^m = \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, H_* \mathcal{L}(k)_n^m).$$

When $n = 1$, we omit it; when $m = \infty$, we omit it; and we also abbreviate $\mathrm{Ext} = \mathrm{Ext}(0) = \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. The above inductive filtration of $\mathcal{L}(k)_n$ gives rise to the *algebraic transfinite Atiyah–Hirzebruch spectral sequence* (ATAHSS)

$$\mathrm{Ext}\{\bar{Q}^J : J \text{ CU}, n \leq e(J)\} \Rightarrow \bigoplus_{k\geq 0} \mathrm{Ext}(k)_n.$$

All of these ATAHSSs are determined by the ATAHSS in the case $n = 1$. More precisely, there is a suspension map $L(k) \to L(k)_n$ inducing a surjection

$$E\colon \mathcal{L}(k) \to \mathcal{L}(k)_n, \qquad E(\bar{Q}^J) = \begin{cases} \bar{Q}^J & e(J) \geq n; \\ 0 & e(J) < n. \end{cases}$$

This passes to a map of ATAHSSs which is a surjection on the $E_1$ page, and all differentials in the ATAHSS for $\mathrm{Ext}(k)_n$ lift to differentials in the ATAHSS for $\mathrm{Ext}(k)$. So we focus on the latter. These fit into a square

$$\begin{array}{ccc} \mathrm{Ext}\{\bar{Q}^J : J \text{ CU}\} & \xRightarrow{\text{ASS}} & \pi_* S\{[J] : J \text{ CU}\} \\ \Big\Downarrow{\scriptstyle \text{ATAHSS}} & & \Big\Downarrow{\scriptstyle \text{TAHSS}} \\ \bigoplus_{k\geq 0} \mathrm{Ext}(k) & \xRightarrow{\text{ASS}} & \bigoplus_{k\geq 0} \pi_* L(k) \end{array},$$

[1] Our $\bar{Q}^J$ is really Behrens' $[J]$.

where the horizontal spectral sequences are Adams spectral sequences. This diagram of spectral sequence should "commute" in some suitable sense, and differentials in the ATAHSS heuristically correspond to differentials in the TAHSS which are detected in homology. The $k = 1$ summand has $L(1) = P_1^\infty$, and this approach to understanding $\pi_* P_1^\infty$ goes back to Mahowald [Mah67].

0.4. **The lambda complex.** Let $\mathcal{L} = \bigoplus_{k \geq 0} \mathcal{L}(k)$, and define

$$\mathrm{Sq}_*^r \colon \mathcal{L} \to \mathcal{L}$$

by asking that

$$\bar{Q}^r \bar{Q}^s = \sum_{0 \leq l \leq r-s-2} \binom{2s-r+1+2l}{l} \bar{Q}^{2s+1+l} \bar{Q}^{r-s-1-l} \text{ if } r \leq 2s, \text{ otherwise } \bar{Q}^r \bar{Q}^s$$

$$\mathrm{Sq}_*^r \bar{Q}^n = \sum_{0 \leq j \leq r/2} \binom{n-r}{r-2j} \bar{Q}^{n-r+j} \mathrm{Sq}_*^j$$

$$\mathrm{Sq}_*^r(1) = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases}.$$

The interpretation is that to compute $\mathrm{Sq}_*^r(\bar{Q}^J)$, one uses the first two relations to write $\mathrm{Sq}_*^r(\bar{Q}^J) = \sum_\alpha \bar{Q}^{J_\alpha} \mathrm{Sq}_*^{r_\alpha}$ with $J_\alpha$ CU, then throws out all terms with $r_\alpha \neq 0$.

Now let $\Lambda$ denote the mod 2 lambda algebra. This is the differential graded algebra generated by symbols $\lambda_r$ for $r \geq 0$, subject to

$$\lambda_s \lambda_{2s+r+1} = \sum_{0 \leq i < r/2} \binom{r-1-i}{i} \lambda_{r+s-i} \lambda_{2s+i+1}, \qquad \delta(\lambda_n) = \sum_{1 \leq i \leq n/2} \binom{n-i}{i} \lambda_{n-i} \lambda_{i-1},$$

and with basis

$$\Lambda = \mathbb{F}_2\{\lambda_I : I \text{ admissible}\}.$$

The lambda complex for $\mathcal{L}$ is the differential graded right $\Lambda$-module $\mathcal{L} \otimes \Lambda$, with differential

$$\delta(\bar{Q}^J \otimes 1) = \sum_{r \geq 1} \mathrm{Sq}_*^r(\bar{Q}^J) \otimes \lambda_{r-1}.$$

This satisfies

$$H_*(\mathcal{L}(k) \otimes \Lambda) \cong \mathrm{Ext}(k).$$

In particular, $H_*(\Lambda) \cong \mathrm{Ext}$.

0.5. **The Curtis algorithm.** Note that $\mathcal{L} \otimes \Lambda$ has basis consisting of $\bar{Q}^J \lambda_I$ where $J$ is CU and $I$ is admissible. Define an ordering on this basis by declaring

$$\bar{Q}^J \lambda_I < \bar{Q}^{J'} \lambda_{I'} \text{ if } J < J' \text{ or } J = J' \text{ and } I < I'.$$

0.5.1. **Definition.** Say that $\bar{Q}^J \lambda_I$ *tags* $\bar{Q}^{J'} \lambda_{I'}$, denoted $\bar{Q}_{J'} \lambda_{I'} \leftarrow \bar{Q}_J \lambda_I$, if $\bar{Q}^J \lambda_I$ is minimal among basis elements for which there exists some $p, q \in \mathcal{L} \otimes \Lambda$ with leading terms $\bar{Q}^J \lambda_I$ and $\bar{Q}^{J'} \lambda_{I'}$ and satisfying $\delta(p) = q$. ◁

For now, let us motivate this relation by noting the following.

0.5.2. **Proposition.** $\bigoplus_{k \geq 0} \mathrm{Ext}(k)$ has a basis indexed by those elements $\bar{Q}^J \lambda_I$ which do not participate in a tag. □

There is a standard algorithm for computing this tag relation. First, to simplify notation, note that the tag relation may be defined whenever one has a map between vector spaces over $\mathbb{F}_2$ with specified well-ordered bases. Our basis for $\mathcal{L}\otimes\Lambda$ is in fact not well-ordered, but the tag relation is local, and $\mathcal{L}\otimes\Lambda$ is locally finite-dimensional, so there is no issue.

In general, fix a map $f\colon \mathbb{F}_2\{x_1,\ldots\}\to\mathbb{F}_2\{y_1,\ldots\}$. Let $k\geq 1$, and suppose that for all $i<k$ we have determined what $x_i$ tags, if anything. We then "process" $x_k$.

An element $p\in\mathbb{F}_2\{x_1,\ldots\}$ with leading term $x_k$ is processed as follows. If $d(p)=0$, then we add $x_k$ to our listing of basis elements which do not tag anything, and move on to $x_{k+1}$. So suppose $d(p)=q\neq 0$, and let $y_l$ be the leading term of $q$. If $y_l\leftarrow x_i$ for some $i<k$, then we process $p+x_i$. Otherwise, we add $y_l\leftarrow x_k$ to our list of tags, and move on to $x_{k+1}$.

Call this the *basic homology algorithm*. The basic homology algorithm is simple, but $\mathcal{L}\otimes\Lambda$ is far too large for it to be useful directly.

Consider the classical case of just $\Lambda$. This is filtered by subcomplexes

$$\Lambda(u)=\mathbb{F}_2\{\lambda_I : I=(i_1,\ldots,i_n)\text{ admissible, } i_1<u\},$$

which fit into short exact sequences

$$0\to\Lambda(u)\to\Lambda(u+1)\to\Lambda(2u+1)\to 0,$$

where the first map is the inclusion and the second map is given by

$$\lambda_{i_1}\cdots\lambda_{i_n}\mapsto\begin{cases}\lambda_{i_2}\cdots\lambda_{i_n} & i_1=u;\\ 0 & i_1\neq u.\end{cases}$$

These maps are compatible with the ordering of basis elements, and so one obtains the following *propagation rule*: $\lambda_{I'}\leftarrow\lambda_I$ if and only if $\lambda_u\lambda_{I'}\leftarrow\lambda_u\lambda_I$, provided $\lambda_u\lambda_I$ and $\lambda_u\lambda_{I'}$ are admissible. The *Curtis table* is the listing of tags $\lambda_{I'}\leftarrow\lambda_I$ such that $I'$ and $I$ have distinct initial terms, together with all basis elements not appearing in a tag. This table is now small enough to be understandable, and the propagation rule allows one to read off the entire tag relation from the Curtis table.

A *Curtis algorithm* is any refinement of the basic homology algorithm which takes into account at least this propagation rule. One may adapt this story to $\mathcal{L}\otimes\Lambda$, using the short exact sequences induced by Eq. (1). This yields the following propagation rules.

0.5.3. **Proposition.** The following hold.

(1) $\bar{Q}^J\lambda_I\leftarrow\bar{Q}^{J'}\lambda_{I'}\Leftrightarrow\bar{Q}^J\bar{Q}^u\lambda_I\leftarrow\bar{Q}^{J'}\bar{Q}^u\lambda_{I'}$, provided $\bar{Q}^J\bar{Q}^u$ and $\bar{Q}^{J'}\bar{Q}^u$ are CU.

(2) In particular, $\bar{Q}^J\lambda_{I'}\leftarrow\bar{Q}^J\lambda_I$ if and only if $\lambda_{I'}\leftarrow\lambda_{I'}$ in the classical Curtis table. □

Define the *Goodwillie Curtis table* to be the classical Curtis table, together with the listing of all tags $\bar{Q}^J\lambda_I\leftarrow\bar{Q}^{J'}\lambda_{I'}$ with $e(J)\neq e(J')$, as well as all additional basis elements not appearing in a tag. This table should be small enough to be understandable, and the above discussion explains how one may read off the entire tag relation from the Goodwillie Curtis table.

Moreover, the basic homology algorithm for $H_*(\mathcal{L}\otimes\Lambda)$ may be augmented by the above propagation rules to yield an algorithm for computing the tag relation for $\mathcal{L}\otimes\Lambda$ which should be effective, at least in a range. In short, we may take as input a classical Curtis table. Once again we process the elements $\bar{Q}^J\lambda_I$ in some suitable increasing order, only now with at least the following shortcuts. First, by (2), we need only consider elements $\bar{Q}^J\lambda_I$ such that $\lambda_I$ does not appear in a tag in the classical Curtis table, i.e. such that $\lambda_I$ corresponds to an element of Ext. Second, by (1), if we already know $\bar{Q}^{J'}\lambda_{I'}\leftarrow\bar{Q}^J\lambda_I$, then we need not process any basis element of the form $\bar{Q}^J\bar{Q}^K\lambda_I$ such that $\bar{Q}^{J'}\bar{Q}^K$ is CU.

Finally we come to the real significance of the Goodwillie Curtis table. The classical Curtis table is more than just a way of packaging a computation of $H_*(\Lambda)$: it also contains within it a computation of each $H_*(\Lambda(u))$, as well as information about the differentials in the spectral sequence associated to the filtration $\Lambda \cong \operatorname{colim}_{u\to\infty} \Lambda(u)$. Once again, the same story may be carried out for $\mathcal{L} \otimes \Lambda$.

Note that Proposition 0.5.3 implies that if $\bar{Q}^{J'}\lambda_{I'} \leftarrow \bar{Q}^J\lambda_I$ with $J \neq J'$, then $\lambda_{I'}$ and $\lambda_I$ do not appear in a tag in the classical Curtis table, and therefore correspond to elements of Ext. The relation between tags and spectral sequences in our context yields the following.

0.5.4. **Theorem.** The following hold.

1. (1) There is a basis for $\mathrm{Ext}(k)_n^m$ indexed by those $\bar{Q}^J\lambda_I \in \mathcal{L} \otimes \Lambda$ with $|J| = k$ and $n \leq e(J) \leq m$, and for which there is no tag of the form $\bar{Q}^{J'}\lambda_{I'} \leftarrow \bar{Q}^J\lambda_I$ or $\bar{Q}^J\lambda_I \leftarrow \bar{Q}^{J'}\lambda_{I'}$ with $n \leq e(J') \leq m$.
2. (2) Tags $\bar{Q}^{J'}\lambda_{I'} \leftarrow \bar{Q}^J\lambda_I$ with $J' \neq J$, thus with $\lambda_{I'}$ and $\lambda_I$ associated to elements of Ext, are in correspondence with differentials $d(\bar{Q}^{J'}\lambda_{I'}) = \bar{Q}^J\lambda_I$ in the ATAHSS. □

Thus the Goodwillie Curtis table contains exactly the data of the ATAHSS for $\bigoplus_{k\geq 0} \mathrm{Ext}(k)$. Such a table is contained in the accompanying goodwillie-curtis-table.txt. The $k = 1$ portion of this table was scraped from [WX16] instead of recomputed.

## References

[Beh12] Mark Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.

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