$\mathbb{R}$-MOTIVIC $K(1)$-LOCALIZATION

WILLIAM BALDERRAMA

We work in the cellular $\mathbb{R}$-motivic category. Let KQ and KGL denote the $\mathbb{R}$-motivic spectra of Hermitian and algebraic K-theory, related by $KGL \simeq KQ/(\eta)$. See [BH21, Section 3] (where $KQ = KO$) for background. For $X, Y \in \mathcal{S}\mathrm{p}_{\mathbb{R}}$, write

$$X \mathbin{\widehat{\otimes}} Y = (X \otimes Y)^{\wedge}_{(2,\eta)}.$$

Fix a cellular $\mathbb{R}$-motivic spectrum X, and define

$$LX = \mathrm{Fib}\left(\psi^3 - 1 \colon X \mathbin{\widehat{\otimes}} KQ \to X \mathbin{\widehat{\otimes}} KQ\right).$$

0.0.1. **Theorem.** There is an equivalence $L_{KGL/(2)}X \simeq LX$. ◁

The main input to this theorem is the following.

0.0.2. **Lemma.** Let Y be a cellular $\mathbb{C}$-motivic spectrum. Then there is a fiber sequence

$$Y \otimes KGL/(2) \longrightarrow Y \otimes KGL/(2) \mathbin{\widehat{\otimes}} KQ \xrightarrow{\psi^3-1} Y \mathbin{\widehat{\otimes}} KGL/(2) \mathbin{\widehat{\otimes}} KQ \ , \tag{1}$$

where for this lemma only KGL and KQ refer to their $\mathbb{C}$-motivic forms.

Proof. As $KGL/(2) = KQ/(2,\eta)$, the completed tensor products in Eq. (1) are ordinary tensor products. Thus the sequence is stable under colimits, and by cellularity we may reduce to the case where $Y = S^{2a,a}$ for some $a \in \mathbb{Z}$. We now appeal to the relation between $\mathbb{C}$-motivic homotopy theory and BP-synthetic homotopy theory. With notation from [GIKR18], we have

$$S^{2a,a} \simeq \Gamma_\star(S^{2a}), \qquad KGL \simeq \Gamma_\star(KU), \qquad KQ \simeq \Gamma_\star(KO).$$

Consider the classical cofiber sequence

$$S^{2a} \otimes KU/(2) \longrightarrow S^{2a} \otimes KU/(2) \otimes KO \xrightarrow{\psi^3-1} S^{2a} \otimes KU/(2) \otimes KO \ . \tag{2}$$

As KU and S^{2a} are filtered colimits of even finite spectra [HS99, Proposition 2.12], we have

$$\Gamma_\star(S^{2a} \otimes KU \otimes KO) \simeq S^{2a,a} \otimes KGL \otimes KQ.$$

By [GIKR18, Proposition 3.17], $\Gamma_\star$ preserves the cofiber of 2 on this, yielding

$$\Gamma_\star(S^{2a} \otimes KU/(2) \otimes KO) \simeq S^{2a,a} \otimes KGL/(2) \otimes KQ.$$

Likewise we have

$$\Gamma_\star(S^{2a} \otimes KU/(2)) \simeq S^{2a,a} \otimes KGL/(2).$$

Thus we learn that applying $\Gamma_\star$ to Eq. (2) yields Eq. (1), and this remains a fiber sequence by [GIKR18, Proposition 3.17], yielding the lemma. □

The following is essentially a formal property of localizations with respect to a quotient.

0.0.3. **Lemma.** LX is $KGL/(2)$-local.

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Proof. As $KGL/(2)$-local objects are closed under fibers, it suffices to show that $X \mathbin{\widehat{\otimes}} KQ$ is $KGL/(2)$-local. Let C be $KGL/(2)$-acyclic; we must show that $[C, X \mathbin{\widehat{\otimes}} KQ] = 0$. As $KGL/(2) = KQ/(2,\eta)$, it follows that C is $KQ/(2^i,\eta^j)$-acyclic for all $i,j \geq 1$. Thus

$$\mathcal{S}\mathrm{p}_{\mathbb{R}}(C, X \mathbin{\widehat{\otimes}} KQ) \simeq \lim_{i,j} \mathcal{S}\mathrm{p}_{\mathbb{R}}(\mathcal{S}\mathrm{p}_{\mathbb{R}}(C, KQ/(2^i,\eta^j)), X) \simeq 0. \qquad \square$$

We can now give the following.

Proof of Theorem 0.0.1. We must show that LX is $KGL/(2)$-local and that $X \to LX$ is a $KGL/(2)$-equivalence. The first follows from Lemma 0.0.3, so it suffices to show the latter. Equivalently, we must verify that

$$KGL/(2) \otimes X \longrightarrow KGL/(2) \otimes X \mathbin{\widehat{\otimes}} KQ \xrightarrow{\psi^3 - 1} KGL/(2) \otimes X \mathbin{\widehat{\otimes}} KQ \tag{3}$$

is a fiber sequence. By the ρ-fracture square, it suffices to check this after real realization (inverting ρ) and after base change to $\mathbb{C}$ (killing ρ).

Recall that the real realization of $KGL/(2)$ vanishes [BH21, Lemma 3.9]. As $KGL/(2) = KO/(2,\eta)$, the completed tensor products in Eq. (3) are ordinary tensor products. As real realization is strong monoidal, it follows that all terms Eq. (3) are sent to 0 by real realization, so the real realization of Eq. (3) is indeed a fiber sequence.

So we have reduced to verifying that Eq. (3) is a fiber sequence after base change to $\mathbb{C}$. As base change preserves cellular objects, this follows from Lemma 0.0.2. $\square$

References

[BH21] Tom Bachmann and Michael J. Hopkins. η-periodic motivic stable homotopy theory over fields. *arXiv e-prints*, page arXiv:2005.06778, May 2021.

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