$\mathbb{F}_p$-SYNTHETIC p-PROFINITE SPACES

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Fix a positive prime p and let $\mathcal{G}\mathrm{em}_p \subseteq \mathcal{G}\mathrm{pd}_\infty$ be the full subcategory spanned by finite products of the Eilenberg–MacLane spaces $K(\mathbb{F}_p, n)$ for $n \geq 0$. In [Bal21, Example 1.3.9], I observed that $\mathcal{G}\mathrm{em}_p^{\mathrm{op}}$ is a finitary loop theory and asserted that $\mathcal{M}\mathrm{odel}^{\mathrm{op}}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}}$ is a reasonable category of $\mathbb{F}_p$-synthetic p-profinite spaces, but did not include a proof of any of my claims, in particular of the identification of the special and generic fibers. A few people have asked about this, so this note fills in some of the details.

We begin by identifying the special fiber. Write $\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$ for the category of unstable rings over the Steenrod algebra.

0.1. **Proposition.** There is an equivalence $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}} \simeq \mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$.

Proof. In a sense, this is just the definition of a $\mathcal{U}$-ring. By definition, a $\mathcal{U}$-ring is a graded set together with various zero-ary, unary, and binary operations, all subject to certain universally quantified axioms. In other words, $\mathcal{U}$-rings are models for a multisorted algebraic theory, and thus $\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\mathrm{free}}}$.

As the cohomology of any space is naturally a $\mathcal{U}$-ring, cohomology defines a functor

$$H^*(-;\mathbb{F}_p)\colon \mathcal{G}\mathrm{em}_p^{\mathrm{op}} \to \mathcal{R}\mathrm{ing}_{\mathcal{U}}. \tag{1}$$

The axioms of a $\mathcal{U}$-ring are cooked up precisely so that Cartan's computation of the cohomology of Eilenberg–MacLane spaces proves that $H^*K(\mathbb{F}_p, n)$ is the free $\mathcal{U}$-ring on a generator in cohomological degree n, and this can be upgraded to show that Eq. (1) restricts to an equivalence $\mathrm{h}\mathcal{G}\mathrm{em}_p^{\mathrm{op}} \simeq \mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\mathrm{free}}$. Thus

$$\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\mathrm{free}}} \simeq \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}}$$

as claimed. ∎

A little more work is required to identify the generic fiber. It is plausible this could be done directly, but the approach I will give instead goes through a synthetic version of Mandell's p-adic homotopy theory [Man01, Lur11] (see [Lur07] for a very readable exposition).

0.2. **Lemma.** Let $\mathcal{P}$ and $\mathcal{P}'$ be loop theories, and let $f\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'}$ be a functor. Suppose that the induced functor $\overline{f}_!\colon \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'}$ is fully faithful and preserves finite limits.[1] Then $f_!\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ is fully faithful and preserves loop models.

Proof. We first verify that $f_!$ is fully faithful. Fix $X, Y \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, and consider the map $\mathrm{Map}_{\mathcal{P}}(X,Y) \to \mathrm{Map}_{\mathcal{P}'}(f_!X, f_!Y)$. It is possible to show that $f_!\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ is, in a suitably precise sense, compatible with the construction of derived Postnikov towers in the

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[1] Here I write $\overline{f}_!$ for the derived functor, denoted $\mathbb{L}\overline{f}$ in [Bal21].

sense of [Bal21, Section 5.4],[2] and so by induction up the derived Postnikov tower of Y it suffices to verify that the induced maps

$$\mathrm{Map}_{\mathrm{h}\mathcal{P}}(\tau_! X, B^{n+1}_{\tau_! Y}\tau_! Y_{S^n}) \to \mathrm{Map}_{\mathrm{h}\mathcal{P}'}(\tau_! f_! X, B^{n+1}_{\tau_! f_! Y}\tau_! f_! Y_{S^n}) \tag{2}$$

on layers are equivalences. As $\overline{f}_!$ preserves geometric realizations and finite limits, we can identify

$$B^{n+1}_{\tau_! f_! Y}\tau_! f_! Y_{S^n} \simeq B^{n+1}_{\overline{f}_!\tau_! Y}\overline{f}_!\tau_! Y_{S^n} \simeq \overline{f}_! B^{n+1}_{\tau_! Y}\tau_! Y_{S^n},$$

and Eq. (2) is just induced by functoriality of $\overline{f}_! \colon \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'}$. As $\overline{f}_!$ is fully faithful, it is thus an equivalence as claimed.

We next verify that $f_!$ preserves loop models. Let $X \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$, so that wish to prove $f_! X \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'}$. By [Bal21, Corollary 3.2.2], it is equivalent to show that $\tau_! f_!(X)$ is 0-truncated. As X is itself a loop model, we can identify

$$\tau_! f_! X \simeq \overline{f}_! \tau_! X \simeq \overline{f}_!(\pi_0 X),$$

so it suffices to verify that $\overline{f}_!$ preserves 0-truncated objects. This follows from the assumption that $\overline{f}_!$ preserves finite limits [Lur17, Proposition 5.5.6.16]. □

Now let $\kappa = \overline{\mathbb{F}}_p$, and write $\mathcal{C}\mathrm{Alg}_\kappa$ for the category of $\mathbb{E}_\infty$ rings over κ and $c \colon \mathcal{G}\mathrm{em}_p^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_\kappa$ for the functor of cochains. This preserves coproducts, and therefore extends to a colimit-preserving functor

$$c_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}} \to \mathcal{M}\mathrm{odel}_{\mathcal{C}\mathrm{Alg}_\kappa^{\mathrm{free}}}.$$

0.3. **Proposition.** The functor $c_!$ is fully faithful and preserves loop models.

Proof. We apply Lemma 0.2. With notation from [Bal23, Section 5], the functor $\overline{c}_!$ decomposes as a composite

$$\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}} \to \mathcal{R}\mathrm{ing}_{\kappa\otimes\mathrm{DL}},$$

where

1. $\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ is restriction along a map of theories derived from the fact that a $\mathcal{U}$-ring is exactly a DL-ring on which Q^0 acts by the identity;
2. $\mathcal{R}\mathrm{ing}_{\mathrm{DL}} \to \mathcal{R}\mathrm{ing}_{\kappa\otimes\mathrm{DL}}$ is given, on underlying modules, by tensoring with κ.

It follows immediately from this decomposition that $\overline{c}_!$ preserves finite limits, and Mandell's work shows that it is fully faithful [Bal23, Proposition 5.3.3]. □

We can now give the promised identification of the generic fiber. Let $\mathcal{F}\mathrm{in}_p$ denote the category of p-finite spaces, and write $i \colon \mathcal{G}\mathrm{em}_p^{\mathrm{op}} \to \mathcal{F}\mathrm{in}_p^{\mathrm{op}}$ for the inclusion. This preserves coproducts and S^n-tensors, and therefore extends to a colimit-preserving functor

$$i_! \colon \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}} \to \mathrm{Ind}(\mathcal{F}\mathrm{in}_p^{\mathrm{op}}).$$

0.4. **Theorem.** The functor $i_!$ is an equivalence of categories.

Proof. First we verify that $i_!$ is fully faithful. Consider the diagram

$$\begin{array}{ccccc} \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}} & \rightarrowtail & \mathcal{M}\mathrm{odel}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}} & \overset{c_!}{\rightarrowtail} & \mathcal{M}\mathrm{odel}_{\mathcal{C}\mathrm{Alg}_\kappa^{\mathrm{free}}} \\ \downarrow{\scriptstyle i_!} & & \overset{\tilde{c}_!}{\searrow} & & \uparrow \\ \mathrm{Ind}(\mathcal{F}\mathrm{in}_p^{\mathrm{op}}) & \xrightarrow{C^\bullet(-;\kappa)} & \mathcal{C}\mathrm{Alg}_\kappa & \overset{\simeq}{\rightarrowtail} & \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{C}\mathrm{Alg}_\kappa^{\mathrm{free}}} \end{array}.$$

[2] A careful formulation will appear in forthcoming work with Piotr Pstrągowski on synthetic spaces.

All functors in the outer rectangle, except possibly $i_!$, are fully faithful: $c_!$ by the first half of Proposition 0.3 and $C^\bullet(-;\kappa)$ by [Lur11, Proposition 3.1.16]. To show that $i_!$ is fully faithful, it therefore suffices to verify that the diagram in fact commutes. By the second half of Proposition 0.3, the top horizontal composite lands in the full subcategory $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_\kappa} \subseteq \mathcal{M}\mathrm{odel}_{\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_\kappa}$, giving the indicated lifting $\tilde{c}_!$ for which the top triangle commutes. As $c_!$ preserves colimits, so does $\tilde{c}_!$. As all functors in the bottom triangle preserve colimits, to verify that it commutes it suffices to verify that it commutes after restriction to $\mathcal{G}\mathrm{em}_p^{\mathrm{op}}$. This now just follows from the definitions of the functors involved.

Next we verify that $i_!$ is essentially surjective. As $i_!$ is colimit-preserving and fully faithful, its essential image is closed under colimits. As $\mathcal{F}\mathrm{in}_p$ is generated under finite limits by $\mathcal{G}\mathrm{em}_p$, it follows that $\mathrm{Ind}(\mathcal{F}\mathrm{in}_p^{\mathrm{op}})$ is generated under colimits by $\mathcal{G}\mathrm{em}_p^{\mathrm{op}}$. As the essential image of $i_!$ contains $\mathcal{G}\mathrm{em}_p^{\mathrm{op}}$ by construction, it follows that $i_!$ is essentially surjective.

As $i_!$ is fully faithful and essentially surjective, it is an equivalence of categories. □

0.5. **Corollary.** There is an equivalence $(\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{G}\mathrm{em}_p^{\mathrm{op}}})^{\mathrm{op}} \simeq \mathrm{Pro}(\mathcal{F}\mathrm{in}_p)$. □

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