UNSTABLE REALIZATIONS OF h_j

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Abstract. We classify for which n the unstable $\mathcal{A}$-algebra functor applied to a nontrivial extension of unstable $\mathcal{A}$-modules $\Sigma^n\mathbb{F}_2 \to M \to \Sigma^{n+2^j}\mathbb{F}_2$ is realizable as the mod 2 cohomology of a fiber sequence. We explain how this problem is equivalent to the classical question of when the Whitehead product $[\iota_n, \alpha]$ vanishes for α the suspension of a Hopf class, studied and resolved in classical work by many people. We then give a streamlined resolution of this classical problem using the unstable Adams spectral sequence.

1. Introduction

We work at the prime 2, and all cohomology is with mod 2 coefficients. Let $\mathcal{A}$ denote the Steenrod algebra and $\mathcal{U}$ the category of unstable $\mathcal{A}$-modules. For $M \in \mathcal{U}$, write $U(M)$ for the free unstable $\mathcal{A}$-algebra on M. We are interested in the following realizability problem.

1.1. **Problem.** Given a short exact sequence

$$0 \to K \to M \to C \to 0 \tag{1}$$

in $\mathcal{U}$, when does there exist a 2-primary fiber sequence

$$B \leftarrow E \leftarrow F$$

satisfying

$$H^*(B \leftarrow E \leftarrow F) \cong U(K \to M \to C)?$$

In this case, we say that the class $e \in \mathrm{Ext}^1_{\mathcal{U}}(C, K)$ classifying Eq. (1) is *U-realizable.*

Our interest in this problem arose from ongoing work with Francis Baer, Eva Belmont, and Dan Isaksen to compute with the unstable Adams spectral sequence: in the situation of Problem 1.1, the Yoneda composition

$$e \circ (-)\colon\ \mathrm{Ext}^*_{\mathcal{U}}(K, -) \to \mathrm{Ext}^{*+1}_{\mathcal{U}}(C, -)$$

participates a map of spectral sequences detecting the boundary map $\Omega F \to B$. We are, to start, particularly interested in the case where F and B are spheres. Recall that

$$\mathrm{Ext}^1_{\mathcal{U}}(\Sigma^n\mathbb{F}_2, \Sigma^{n+s}\mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{h_j^{(n)}\} & \text{if } n \geq s = 2^j, \\ 0 & \text{otherwise,} \end{cases}$$

with nonzero class $h_j^{(n)} \in \mathrm{Ext}^1_{\mathcal{U}}(\Sigma^n\mathbb{F}_2, \Sigma^{n+2^j}\mathbb{F}_2)$ classifying the extension

$$0 \to \Sigma^{n+2^j}\mathbb{F}_2 \to H_j^{(n)} \to \Sigma^n\mathbb{F}_2 \to 0$$

with nontrivial action of Sq^{2^j}. We shall prove the following.

1.2. **Theorem.** $h_j^{(n)} \in \mathrm{Ext}^1_{\mathcal{U}}(\Sigma^n\mathbb{F}_2, \Sigma^{n+2^j}\mathbb{F}_2)$ is U-realizable if and only if $j \leq 3$ and:

(0) If $j = 0$, then $n \equiv -1 \pmod 2$;
(1) If $j = 1$, then $n \equiv -1 \pmod 4$ or $n \in \{2, 6\}$;
(2) If $j = 2$, then $n \equiv -1 \pmod 8$ or $n = 2^k - 3$ for some $k \geq 3$;
(3) If $j = 3$, then $n \equiv -1 \pmod{16}$ or $n = 11$. □

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1.3. **Example.** For $j \le 3$ and $n \equiv -1 \pmod{2^{j+1}}$, the class $h_j^{(n)}$ is U-realized by the Stiefel fibrations

$$\begin{aligned} S^{2k-1} &\longrightarrow V_2(\mathbb{R}^{2k+1}) \longrightarrow S^{2k} \\ S^{4k-1} &\longrightarrow V_2(\mathbb{C}^{2k+1}) \longrightarrow S^{4k+1} \\ S^{8k-1} &\longrightarrow V_2(\mathbb{H}^{2k+1}) \longrightarrow S^{8k+3} \\ S^{16k-1} &\longrightarrow V_2(\mathbb{O}^{2k+1}) \longrightarrow S^{16k+7}. \end{aligned}$$

For the first three this is standard, and we elaborate on the fourth in Example 2.2. The extension $h_2^{(2)}$ is realized by the fibration

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1$$

sending a complex line $L \subset \mathbb{C}^4 \cong \mathbb{H}^2$ to the quaternionic line $\mathbb{H} \otimes_{\mathbb{C}} L \subset \mathbb{H}^2$.[1] We do not have models for the remaining fibrations, although it seems likely that one exists for $h_1^{(6)}$. The extension $h_3^{(15)}$ may also be U-realized by a 2-primary fibration

$$S^{15} \longrightarrow F_4/G_2 \longrightarrow S^{23}$$

constructed by homotopical methods in [DM91] and known to not exist integrally.

As we shall explain, standard considerations reduce Theorem 1.2 to the following.

1.4. **Theorem.** Let $\alpha \in \pi_{n+2^j-1}S^n$ be a suspension of the jth Hopf class, defined for $j \le 3$ and $n \ge 2^j$. Then the Whitehead product $[\iota_n, \alpha] \in \pi_{2n+2^j-2}S^n_{(2)}$ vanishes for exactly the pairs (j, n) described in Theorem 1.2. $\square$

This theorem is essentially known due to work of Hopf, Whitehead, Hilton, Toda, Adams, Barratt, Mahowald, Kristensen, Madsen, etc.; see especially [Hop35, HW53, Hil55, Mah65, KM67, Mah77]. As enumerated there, the only cases unresolved at the time of [KM67] were $[\iota_{2^k-3}, \nu]$ for $k \ge 5$ and $[\iota_{27}, \sigma]$; the former was resolved in [Mah77], and it seems likely the latter was known around the time of [MT67], though we have not seen it stated explicitly.

The point of this note is twofold. First, to provide a reference for Theorem 1.2. Second, to give a streamlined account of Theorem 1.4, the literature for which is somewhat spread out.

1.5. **Warning.** U-realizability of the extension $h_j^{(n)}$ is distinct from U-realizability of the module $H_j^{(n)}$, i.e. existence of a space X for which $H^*(X) \cong U(H_j^{(n)})$. In particular:

(1) The class $h_2^{(4)}$ is not U-realizable despite $H^*(\mathbb{H}P^3) \cong U(H_2^{(4)})$; note $\mathbb{H}P^3/S^4 \simeq C(2\nu)$.
(2) Work of Gonçalves [Gc78, Proof of Corollary 1.3] implies that $H_3^{(8)}$ is not U-realizable, but this does not follow from our proof that $h_3^{(8)}$ is not U-realizable.

2. Basic reductions

We start by reducing Theorem 1.2 to Theorem 1.4.

2.1. **Lemma.** $h_j^{(n)}$ is U-realizable if and only if there exists a 2-primary spherical fibration

$$S^n \to E \to S^{n+2^j}$$

[1]Thanks to Christian Kremer for pointing out this construction.

with the property that the composite

$$\alpha\colon S^{n+2^j-1} \to \Omega S^{n+2^j} \to S^n$$

of the boundary map with the inclusion of the bottom cell is detected by h_j.

Proof. As $\Sigma^n \mathbb{F}_2$ is realized uniquely up to $\mathbb{F}_2$-equivalence by S^n, a U-realization of $h_j^{(n)}$ is given by such a fiber sequence with

$$\widetilde{H}^*(E) \cong \mathbb{F}_2\{e_n, e_{n+2^j}, e_{2n+2^j}\}$$

satisfying

$$e_n \cdot e_{n+2^j} = e_{2n+2j}, \quad Sq^{2^n}(e_n) = e_{n+2^j}, \quad Sq^n(e_{n+2^j}) = 0 = Sq^{n+2^j}(e_n)$$

The $(n+2^j)$-skeleton of E is equivalent to the cofiber of α, and so the identity $Sq^{2^n}(e_n) = e_{n+2^j}$ implies that α is detected by h_j.

Conversely, given such a fiber sequence, the Serre spectral sequence implies that $\widetilde{H}^*(E) \cong \mathbb{F}_2\{e_n, e_{n+2^j}, e_{2n+2^j}\}$ with $e_n \cdot e_{n+2^j} = e_{2n+2^j}$. Moreover, if α is detected by h_j then $Sq^{2^n}(e_n) = e_{n+2^j}$. We claim that necessarily $H^*(E) \cong U(H_j^{(n)})$, so that this fiber sequence must be a U-realization of $h_j^{(n)}$.

First suppose $n > 2^j$. It follows that $H^*(E)$ is isomorphic to an exterior algebra $\Lambda(e_n, e_{n+2^j})$. To show that $H^*(E) \cong U(H_j^{(n)})$ we must show that $Sq^{n+2^j}(e_n) = 0$ and $Sq^n(e_{n+2^j}) = 0$. The former follows by instability. The map $E \to S^{n+2^j}$ in cohomology sends the fundamental class of S^{n+2^j} to e_{n+2^j}, and this shows $Sq^n(e_{n+2^j}) = 0$.

Next suppose $n = 2^j$. By the instability condition, $e_{2^j+2^j} = Sq^{2^j}(e_{2^j}) = e_{2^j}^2$, and therefore $H^*(E)$ is isomorphic to a truncated polynomial ring $\mathbb{F}_2[e_{2^j}]/(e_{2^j}^4)$. To show $H^*(E) \cong U(H_j^{(n)})$ we must show that $Sq^{2^j}(e_{2^j}^2) = 0$. This follows from the Cartan formula. □

2.2. **Example.** In [Jam58, Section 8], James constructs octonionic Stiefel fibrations

$$S^{8n-1} \to V_2(\mathbb{O}^{n+1}) \to S^{8n+7}.$$

We claim that if $n = 2k$ then this is a U-realization of $h_3^{(16k-1)}$. Following Lemma 2.1, it suffices to show that the bottom attaching map $\alpha\colon S^{16k+6} \to S^{16k-1}$ of $V_2(\mathbb{O}^{2k+1})$ is detected by h_3. It seems plausible that this could admit a geometric proof, perhaps related to the equivalence $C(\sigma) \simeq \mathbb{O}P^2$, but it may also be verified indirectly as follows.

The Cayley–Dickson construction of the octonions equips the normed algebra $\mathbb{O}$ with an action by the group C_2 for which we may identify $\mathbb{O}^{C_2} \cong \mathbb{H}$. This extends to an action on $V_2(\mathbb{O}^n)$ for which $V_2(\mathbb{O}^n)^{C_2} \cong V_2(\mathbb{H}^n)$. In this way we may regard α as a C_2-equivariant map $\alpha\colon S^{(8k+3)(1+\sigma)} \to S^{8k\sigma+(8k-1)}$ for which α^{C_2} is detected by h_2.

After stablization, α determines a class in the C_2-equivariant stable stem $\pi_{4+3\sigma} S_{C_2}$ satisfying $\Phi^{C_2}(\alpha) = u \cdot \nu$ for a unit u. By [AI82, Theorem 14.18], all such classes are C_2-equivariant lifts of $u' \cdot \sigma$ for a unit u'. Therefore $\alpha\colon S^{16k+6} \to S^{16k-1}$ is detected by h_3 as claimed.

2.3. **Lemma.** Fix $\alpha \in \pi_{n+s} S^n$. Then the Whitehead product $[\iota_n, \alpha] \in \pi_{2n+s-1} S^n$ vanishes if and only if there exists a spherical fibration

$$S^n \to E \to S^{n+s+1}$$

with the property that the composite

$$\alpha\colon S^{n+2^j-1} \to \Omega S^{n+2^j} \to S^{n+s+1}$$

of the boundary map with the inclusion of the bottom cell is homotopic to α.

Proof. By the classification of fiber bundles, fibrations $E \to S^{n+s+1}$ with fibers homotopy equivalent to S^n are in correspondence with homotopy classes of maps

$$S^{n+s+1} \to B\mathrm{Aut}(S^n),$$

where $\mathrm{Aut}(S^n)$ is the space of homotopy automorphisms of S^n. By choosing a basepoint of S^{n+s+1} we may make this into a pointed map, adjoint to a pointed map

$$S^{n+s} \to \mathrm{Aut}(S^n) \subset \mathrm{Map}(S^n, S^n),$$

where $\mathrm{Map}(S^n, S^n)$ is pointed at the identity. By adjunction, such maps are equivalent to maps

$$m\colon S^n \times S^{n+s} \to S^n$$

satisfying $m(-,*) = \iota_n$, and under this adjunction we may identify $\alpha = m(*,-)$.

Thus we have shown that there exists a spherical fibration $S^n \to E \to S^{n+s+1}$ with boundary map α on the bottom cell if and only if there exists a product $m\colon S^n \times S^{n+s} \to S^n$ satisfying $m(-,*) = \iota_n$ and $m(*,-) = \alpha$. Such a product exists if and only if $[\iota_n, \alpha] = 0$. $\square$

The above lemmas together, along with Adams' resolution of the Hopf invariant one problem [Ada60], combine to reduce Theorem 1.2 to Theorem 1.4.

3. Computing the Whitehead product

It remains to determine when $[\iota_n, \alpha]$ vanishes for $\alpha \in \{2, \eta, \nu, \sigma\}$. Whitehead products at the prime 2 are most efficiently computed using the EHΔ sequence

$$\cdots \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\Delta_n} S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H_n} \Omega S^{2n+1} .$$

By work of James [Jam56, Jam57], if $\alpha, \beta \in \pi_* S^n$ then

$$[\alpha, \beta] = \Delta_n(E(\alpha \wedge \beta)).$$

These operations are detected in the unstable Adams spectral sequence that we shall index as

$$U_2^{n,s,f} = \mathrm{Ext}^f_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, \Sigma^{n+s+f}\mathbb{F}_2) \Rightarrow \pi_{n+s} S^n.$$

Specifically, there is an algebraic EHΔ sequence

$$\cdots \longrightarrow U_2^{2n+1,s-n+1,f-2} \xrightarrow{\Delta_n} U_2^{n,s,f} \xrightarrow{E} U_2^{n+1,s,f} \xrightarrow{H_n} U_2^{2n+1,s-n,f-1} \longrightarrow \cdots \quad (2)$$

detecting the topological EHΔ sequence [Cur71], as well as pairings

$$U_2^{n_1,s_1,f_1} \times U_2^{n_2,s_2,f_2} \to U_2^{n_1+n_2,s_1+s_2,f_1+f_2}$$

detecting smash products, suspended from more refined composition pairings [BK73]. Writing generically $h_j \in U_2^{n,2^j-1,1}$ for $n \geq 2^j$, we see that the Whitehead products we are interested in are detected by $\Delta_n(h_j)$ for $j \leq 3$ and $n \geq 2^j$.

The values of $\Delta_n(h_j)$ may be read off a Curtis table,[2] which is a way of organizing $H^*(\mathcal{A})$ as computed via the lambda algebra [Tan85, CGMM87], and is well understood in low filtration going back to Wang's computation of $H^{\leq 3}(\mathcal{A})$ [Wan67]. Specifically, there is a tag

$$\lambda_I \leftarrow \lambda_n \lambda_{2^j-1}$$

[2]See https://williamb.info/lambda/classic-curtis-table.txt for a convenient Curtis table.

in the Curtis table if and only if λ_I is the name of $\Delta_n(h_j)$ in $U_2^{n,n+2^j-2,3}$. It follows that $\Delta_n(h_j) = 0$ if and only if there is no such tag. In this case, there are two possibilities: either there is a tag

$$\lambda_n\lambda_{2^j-1} \leftarrow \lambda_{n+2^j},$$

or else $\lambda_n\lambda_{2^j-1}$ does not participate in any tag and so names a stable class in $H^2(\mathcal{A})$. In either case, $\lambda_n\lambda_{2^j-1}$ names a class in $U_2^{n+1,n+2^j-1,2}$ satisfying $H_n(\lambda_n\lambda_{2^j-1}) = h_j$. Combined with known information about the Curtis table, this discussion shows the following.

3.1. **Lemma.** $\Delta_n(h_j) = 0$ in exactly the following cases:

(1) $n \equiv -1 \pmod{2^{j+1}}$, corresponding to the tags

$$\lambda_{2^{j+1}m-1}\lambda_{2^j-1} \leftarrow \lambda_{2^j(2m+1)-1}$$

for $m \geq 1$;

(2) $n = 2^{m+j} - 2^{j-1} - 1$ for $j \geq 1$ and $m = -1$ or $m \geq 1$ corresponding to the nonzero stable class $h_{j-1}h_{j+m}$ named by $\lambda_{2^{m+j}-2^{j-1}-1}\lambda_{2^j-1}$. □

By the algebraic $EH\Delta$ sequence, the suspension $U_2^{n,s,1} \to U_2^{n+1,s,1}$ on the 1-line is a monomorphism. As a consequence, the only nonzero differentials on the 1-line of the unstable Adams spectral sequence are desuspensions of the Hopf invariant one differentials $d_2(h_{k+1}) = h_0h_k^2$ for $k \geq 2$. As these are stably nontrivial, it follows that in all cases $\Delta_n(h_j) \neq h_0h_k^2$. Therefore if $j \leq 3$ and $\Delta_n(h_j) \neq 0$, then $\Delta_n(h_j)$ is a nonzero permanent cycle in the unstable Adams spectral sequence, detecting the corresponding topological value of Δ_n. Thus we have established the following.

3.2. **Lemma.** We have

(1) $\Delta_n(2) \neq 0$ unless possibly when $n \equiv -1 \pmod 2$;
(2) $\Delta_n(\eta) \neq 0$ unless possibly when $n \equiv -1 \pmod 4$ or $n = 2^k - 2$ for $k \geq 2$;
(3) $\Delta_n(\nu) \neq 0$ unless possibly when $n \equiv -1 \pmod 8$ or $n = 2^k - 3$ for $k \geq 3$;
(4) $\Delta_n(\sigma) \neq 0$ unless possibly when $n \equiv -1 \pmod{16}$ or $n = 2^k - 5$ for $k \geq 4$. □

On the other hand, if $\Delta_n(h_j) = 0$, then it could be that the corresponding topological value of Δ_n is nonzero, detected in higher Adams filtration. The values of $\Delta_{2k-1}(2)$, $\Delta_{4k-1}(\eta)$, $\Delta_{8k-1}(\nu)$, $\Delta_{16k-1}(\sigma)$ and $\Delta_2(\eta)$ vanish for geometric reasons: they are realized by the fibrations described in Example 1.3. We are left with verifying that $\Delta_6(\eta) = 0$ but $\Delta_{2^k-2}(\eta) \neq 0$ for $k \geq 4$; that $\Delta_{2^k-3}(\nu) = 0$ for all $k \geq 3$; and that $\Delta_{11}(\sigma) = 0$ but $\Delta_{2^k-5}(\sigma) \neq 0$ for $k \geq 4$. We verify these in turn, beginning with those values which do vanish.

As the octonionic Stiefel fibrations are not nearly as well understood as the real, complex, and quaternionic Stiefel fibrations, we shall also give an independent proof that $\Delta_{16k-1}(\sigma) = 0$. Analogous arguments may be applied to show that $\Delta_{2k-1}(2)$, $\Delta_{4k-1}(\eta)$, and $\Delta_{8k-1}(\nu)$ vanish without resorting to geometric constructions. Similarly, our proof that $\Delta_6(\eta) = 0$ may be adapted to independently prove that $\Delta_2(\eta) = 0$.

3.3. **Lemma.** We have $\Delta_6(\eta) = 0$.

Proof. This holds as η is the stable Hopf invariant of 2σ. In detail, as h_0h_3 is named by $\lambda_6\lambda_1$ we find that $\Delta_6(h_1) = 0$ is realized by $H_6(h_0h_3) = h_1$. As $U_2^{7,6,*} = 0$ for $* \geq 4$, the class $h_0h_3 \in U_2^{7,7,2}$ is a permanent cycle detecting the class $\sigma' \in \pi_{7+7}S^7$.[3] As $U_2^{13,1,1} = \mathbb{F}_2\{h_1\}$ it follows that the identity $H_6(h_0h_3) = h_1$ lifts to $H_6(\nu') = \eta$, and therefore $\Delta_6(\eta) = 0$. □

[3]This also follows from the fact that $h_0^{(7)}$ is U-realizable, given that $h_3 \in U_2^{7,8,1}$ is a permanent cycle.

3.4. **Lemma.** We have $\Delta_{2^k-3}(\nu) = 0$ for $k \geq 3$.

Proof. This holds as ν is the stable Hopf invariant of the η_k family. In detail, as h_1h_k is named by $\lambda_{2^k-3}\lambda_3$, we find that $\Delta_{2^k-3}(h_2) = 0$ is realized by $H_{2^k-3}(h_1h_k) = h_2$. We claim that $h_1h_k \in U_2^{2^k-2,2^k,2}$ is a permanent cycle. As $U_2^{2^k-3,3,*} = \mathbb{F}_2\{h_2, h_0h_2, h_0^2h_3\}$ it then follows that the identity $H_{2^k-3}(h_1h_k) = h_2$ lifts to $H_{2^k-3}(\eta_k) = u \cdot \nu$ for a unit u, and therefore $\Delta_{2^k-3}(\nu) = 0$.

To see that $h_1h_k \in U_2^{2^k-2,2^k,2}$ is a permanent cycle, we may argue as follows. As h_1h_k is stably a permanent cycle, if $d_r(h_1h_k) = x \in U_2^{2^k-2,2^k-1,2+r}$ then x is stably trivial. Write $E_2^{s,f} = \mathrm{Ext}_{\mathcal{A}}^f(\mathbb{F}_2, \Sigma^{s+f}\mathbb{F}_2)$ for the E_2-page of the stable Adams spectral sequence. We claim that the stabilization $U_2^{2^k-2,2^k-1,*} \to E_2^{2^k-1,*}$ is monic for $* \geq 4$, implying that there is no possible such x. Consider the algebraic $EH\Delta$ sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & U_2^{2^{k+1}-3,2,*-2} & \xrightarrow{\Delta_{2^k-2}} & U_2^{2^k-2,2^k-1,2+r} & \xrightarrow{E_{2^k-2}} & U_2^{2^k-1,2^k-1,2+r} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & U_2^{2^{k+1}-1,1,*-2} & \xrightarrow{\Delta_{2^k-1}} & U_2^{2^k-1,2^k-1,*} & \xrightarrow{E_{2^k-1}} & U_2^{2^k,2^k-1,*} & \longrightarrow & \cdots . \\
\cdots & \longrightarrow & U_2^{2^{k+1}+1,0,*-2} & \xrightarrow{\Delta_{2^k}} & U_2^{2^k,2^k-1,*} & \xrightarrow{E_{2^k}} & U_2^{2^k+1,2^k-1,*} & \longrightarrow & \cdots
\end{array}$$

We have $U_2^{2^{k+1}-3,2,*-2} = 0$ for $* > 4$ and $U_2^{2^{k+1}-3,2,2} = \mathbb{F}_2\{h_1^2\}$. The tag $\lambda_{2^k-2}\lambda_1^2 \leftarrow \lambda_{2^k}\lambda_1$ implies $\Delta_{2^k-2}(h_1^2) = 0$, implying that E_{2^k-2} is monic. Similarly $U_2^{2^{k+1}-1,1,*-2} = 0$ for $* \geq 4$, implying that E_{2^k-1} is monic. In the final case, $U_{2^{k+1}+1,0,f} = \mathbb{F}_2\{h_0^f\}$ for $f \geq 0$. The tag $\lambda_{2^k-1}\lambda_0^{f+1} \leftarrow \lambda_{2^k}\lambda_0^f$ implies that $\Delta_{2^k}(h_0^f)$ is the class named by $\lambda_{2^k-1}\lambda_0^{f+1}$. This class does not desuspend, implying that $E_{2^k} \circ E_{2^k-1}$ is monic even if E_{2^k} is not. As $U_2^{2^k+1,2^k-1,*}$ is in the stable range, altogether this shows that the stabilization $U_2^{2^k-2,2^k-1,*} \to E_2^{2^k-1,*}$ is monic for $* \geq 4$. ☐

3.5. **Lemma.** We have $\Delta_{11}(\sigma) = 0$.

Proof. This holds as σ is the stable Hopf invariant of ν_4. In detail, as h_2h_4 is named by $\lambda_{11}\lambda_7$, we find that $\Delta_{11}(h_3) = 0$ is realized by $H_{11}(h_2h_4) = h_3$. The class $h_2h_4 \in U_2^{12,18,2}$ is stably a permanent cycle, so if $d_r(h_2h_4) = x$ then $x \in U_2^{12,17,2+r}$ is stably trivial. By inspection the stabilization $U_2^{12,17,*} \to E_2^{17,*}$ is monic. Therefore $h_2h_4 \in U_2^{12,18,2}$ is a permanent cycle detecting the class ν_4 on the 12-sphere, and $H_{11}(\nu_4)$ is detected by h_3. As $\pi_{23+7}S^{23} \cong \mathbb{Z}/(16)$ generated by σ, it follows that $H_{11}(\nu_4) = u \cdot \nu$ for a unit u, and therefore $\Delta_{11}(\sigma) = 0$. ☐

3.6. **Lemma.** We have $\Delta_{16k-1}(\sigma) = 0$.

Proof. Let $F_k = \mathrm{Fib}(S^{16k} \to \Omega^8 S^{16k+8})$ and write $i\colon F_k \to S^{16k}$. We claim that there exists a class $\alpha_k \in \pi_{32k+6}F_k$ satisfying $H(i(\alpha_k)) = \sigma$. This is in the metastable range, meaning that the James–Hopf maps provide an isomorphism

$$\pi_{32k+6}F_k \cong \pi_{32k+7}\Sigma^\infty \mathbb{R}P_{16k+1}^{16k+8}$$

for which $H \circ i$ corresponds to projection onto the top cell. In particular by James periodicity there are isomorphisms

$$\pi_{32k+6}F_k \cong \pi_{38}F_1$$

compatible with $H \circ i$, and so it suffices to produce a class $\alpha \in \pi_{16+22}S^{16}$ satisfying $E^8(\alpha) = 0$ and $H(\alpha) = \sigma$.

By the tag $\lambda_{15}\lambda_7 \leftarrow \lambda_{23}$, we find that $\lambda_{15}\lambda_7$ names an element of $U_2^{16,22,2}$ satisfying $E^8(\lambda_{15}\lambda_7) = 0$ and $H(\lambda_{15}\lambda_7) = h_3$. We claim that this detects a class with the desired properties. We first claim that $\lambda_{15}\lambda_7$ is a permanent cycle. As $\lambda_{15}\lambda_7$ stabilizes to $h_4h_3 = 0$, it follows that if $d_r(\lambda_{15}\lambda_7) = y$ then y is stably trivial. For degree reasons the only possible target is a d_2, which would be incompatible with the Hopf invariant one differential on $\lambda_{15} = h_4$. Therefore $\lambda_{15}\lambda_7$ is a permanent cycle.

Fix a class $\alpha \in \pi_{16+22}S^{16}$ detected by $\lambda_{15}\lambda_7$. Then $H(\alpha)$ is detected by $\lambda_7 = h_3$, and as this is in the stable range it follows that $H(\alpha) = u \cdot \sigma$ for some unit u, and by modifying α by a unit we may as well suppose $u = 1$. If $E^8(\alpha) = 0$ then we are done, so suppose that $E^8(\alpha) \neq 0$. This is in the stable range, so the only possible alternative is that $E^8(\alpha) = P\kappa$ is the class detected by Pd_0. This algebraic class desuspends to $U_2^{7,22,8}$ where it is a permanent cycle for degree reasons, implying that $P\kappa$ also desuspends to $\pi_{7+22}S^7$. Therefore if β is a desuspension of $P\kappa$ to S^{16} then $\alpha' = \alpha - \beta$ satisfies $H(\alpha') = H(\alpha) = \sigma$ and $E^8(\alpha') = P\kappa - P\kappa = 0$ as needed. □

It remains only to show that certain values of Δ which vanish in algebra do not vanish in homotopy.

3.7. **Lemma.** We have $\Delta_{2^k-2}(\eta) \neq 0$ for $k \geq 4$ and $\Delta_{2^k-5}(\sigma) \neq 0$ for $k \geq 5$.

Proof. The algebraic identity $\Delta_{2^k-2}(h_2) = 0$ is realized by $H_{2^k-2}(h_0h_k) = h_2$, and the algebraic identity $\Delta_{2^k-5}(h_3) = 0$ is realized by $H_{2^k-5}(h_2h_k) = h_3$. The lemma follows from the Adams differentials on the classes h_0h_k for $k \geq 4$ and h_2h_k for $k \geq 5$, as we now explain.

Consider the stable Adams differential $d_3(h_0h_4) = h_0d_0$. We claim that this desuspends to a differential on S^{15}. The only alternative is that $h_0h_4 \in U_2^{15,15,2}$ supports a nonzero d_2 hitting a stably trivial class, and there are no possible targets. The class $h_0d_0 \in U_2^{15,14,5}$ desuspends to $U_2^{14,14,5}$ (in fact to $U_2^{6,14,5}$). As $U_2^{14,15,\leq 3} = \mathbb{F}_2\{h_0^2h_4\}$ we see that $h_0d_0 \in U_2^{14,14,5}$ is not the target of a differential. Therefore by the geometric boundary theorem [Beh12, Lemma A.4.1(5)], we find that there exists a class $\alpha \in \pi_{30}S^{27}$ detected by h_3 for which $\Delta_{14}(\alpha)$ is detected by h_0d_0. As $\pi_{30}S^{27} \cong \mathbb{Z}/(8)$ generated by σ, necessarily $\alpha = u \cdot \sigma$ for a unit u and therefore $\Delta_{14}(\sigma) \neq 0$ as claimed.

The argument for h_0h_k and h_2h_k with $k \geq 5$ is identical, only using the differentials $d_2(h_0h_k) = h_0^3h_{k-1}^2$ for $k \geq 5$, $d_3(h_2h_5) = h_0p$, and $d_2(h_2h_k) = h_0^2h_2h_{k-1}$ for $k \geq 6$. □

This concludes the proof of Theorem 1.4 and therefore also of Theorem 1.2.

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