Deformations of homotopy theories

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Intro

Big goal

Compute the stable homotopy groups of the sphere spectrum.

Recent breakthrough technique

Compute the stable homotopy groups of *other* sphere spectra.

Form of the technique

1. These other sphere spectra are often *deformations* of $\mathbb{S}$; modules over them give deformations of $\mathcal{S}\mathrm{p}$.
2. Studying these deformations can reveal otherwise hidden information about classical homotopy theory.

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A square in need of a catchy name

The standard context

$$\begin{array}{ccccc}
\mathcal{S}\mathrm{p}_{\mathbb{R}} & \xrightarrow{\mathrm{Be}} & \mathcal{S}\mathrm{p}_{C_2} & \xrightarrow{\Phi} & \mathcal{S}\mathrm{p} \\
{\scriptstyle -\otimes_{\mathbb{R}}\mathbb{C}}\downarrow & & \downarrow{\scriptstyle U} & & \\
\mathcal{S}\mathrm{p}_{\mathbb{C}} & \xrightarrow{\mathrm{Be}} & \mathcal{S}\mathrm{p} & &
\end{array}.$$

Studying these categories together reveals otherwise obscured insights.

No motivic knowledge is needed for this talk.

Some work (highly non-exhaustive)

1. Isaksen-Wang-Xu (2020): use $\mathbb{S}_{\mathbb{C}}$ to compute $\pi_*\mathbb{S}_2$ into 90-stem;
2. Belmont-Isaksen (2020) compute $\pi_*\mathbb{S}_{\mathbb{R}}$, Guillou-Isaksen-? (tbd) compute $\pi_*\mathbb{S}_{C_2}$; these give e.g. root invariants in $\pi_*\mathbb{S}$;
3. Other contexts: Wilson-Østvær (2016) compute $\mathbb{F}_q$-motivic stems, Burklund-(Hahn-Senger,Isaksen-Xu) (2019,tbd): $\mathbb{F}_2$-synthetic π_*;
4. Add your name: there are many more things to compute.

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The deformation viewpoint, or, why is $\mathbb{S}_{\mathbb{C}}$ so useful?

Work at a prime p.

Theorem (Pstrągowski 2018, Gheorghe-Wang-Xu 2018, ...)

There is an element $\tau \in \pi_*\mathbb{S}_{\mathbb{C}}$, and:

1. $\mathbb{S}_{\mathbb{C}}/(\tau)$ is $\mathbb{E}_\infty$, with $\pi_*\mathbb{S}_{\mathbb{C}}/(\tau) = \mathrm{Ext}_{(MU_*, MU_*MU)}(MU_*, MU_*)$ and

$$\mathcal{M}\mathrm{od}_{\mathbb{S}_{\mathbb{C}}/(\tau)} \approx \text{even } MU_*MU\text{-comodules}.$$

2. $\mathbb{S}_{\mathbb{C}}[\tau^{-1}]$ is more easily $\mathbb{E}_\infty$, with $\pi_*\mathbb{S}_{\mathbb{C}}[\tau^{-1}] = \pi_*\mathbb{S} \otimes \mathbb{Z}[\tau^{\pm 1}]$,

$$\mathcal{M}\mathrm{od}_{\mathbb{S}_{\mathbb{C}}[\tau^{-1}]} = \mathcal{S}\mathrm{p}.$$

3. The τ-inverted τ-BSS $\pi_*\mathbb{S}_{\mathbb{C}}/(\tau)[\tau^{\pm 1}] \Rightarrow \pi_*\mathbb{S}[\tau^{\pm 1}]$ is the ANSS.

Thus $\mathcal{S}\mathrm{p}_{\mathbb{C}}^{\mathrm{cell}}$ is a *deformation* with generic fiber $\mathcal{S}\mathrm{p}$, algebraic special fiber, and deformation data = Adams-Novikov data.

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Idea

For computations of stable stems, the motivic stuff is a red herring:

1. Can directly define a "$\mathbb{S}_{MU}$" with $\mathcal{S}\mathrm{p}_{MU} := \mathcal{M}\mathrm{od}_{\mathbb{S}_{MU}} \approx \mathcal{S}\mathrm{p}_{\mathbb{C}}$;
2. Have deformations $\mathbb{S}_R$ and $\mathcal{S}\mathrm{p}_R$ from other R-based Adams SS's;
3. Generally, deformations come anywhere spectral sequences come.

Goal

Explain the filtered object approach to building deformations.

Outline

1. Ordinary filtered algebras as deformations;
2. Spectral sequences and deformations;
3. Various examples.

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Filtered objects: lots of definitions

Fix a category $\mathcal{C}$.

Definition: Filtered objects

The category $\mathcal{C}_{\text{Filt}}$ of *filtered objects* in $\mathcal{C}$ is the category of functors $X\colon (\mathbb{Z}, <) \to \mathcal{C}$:

$$\cdots \to X(-1) \to X(0) \to X(1) \to \cdots .$$

Definition: Day convolution of filtered objects

For $\mathcal{C}$ monoidal with suitable colimits, $\mathcal{C}_{\text{Filt}}$ has monoidal product

$$(X \otimes Y)(n) = \operatorname*{colim}_{p+q \leq n} X(p) \otimes Y(q).$$

Definition: Filtered algebras

Monoids in $\mathcal{C}_{\text{Filt}}$ are *filtered algebras*. E.g. when $\mathcal{C} = \mathcal{M}\text{od}_k$, get filtered k-algebras: $A = \operatorname{colim}_n A_{\leq n}$ with $A_{\leq n} \otimes A_{\leq m} \to A_{\leq n+m}$.

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Filtered objects as graded objects

Observation

There is an equivalence of categories

$$\mathrm{Sy}\colon \mathrm{Fun}(\mathbb{Z}, \mathrm{L}\mathcal{M}\mathrm{od}_k) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{k[\sigma]}, \qquad |\sigma| = 1,$$

where $\mathrm{L}\mathcal{M}\mathrm{od}_{k[\sigma]}$ = graded $k[\sigma]$-modules, given by

$$(\mathrm{Sy}\, M)_n = M_{\leq n}, \qquad \sigma\colon M_{\leq n} \to M_{\leq n+1}.$$

This is sym. monoidal for k commutative (and works more generally).

Filtered algebras as deformations

If $A = \mathrm{colim}_n A_{\leq n}$ is a filtered algebra, then

1. $(\mathrm{Sy}\, A)/(\sigma) = \mathrm{gr}\, A$,
2. $(\mathrm{Sy}\, A)[\sigma^{-1}] = A \otimes \mathbb{Z}[\sigma^{\pm 1}]$.

$\mathrm{Sy}\, A$ is a *deformation* with generic fiber $A[\sigma^{\pm 1}]$ and special fiber $\mathrm{gr}\, A$.

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Categorification

Filtered modules as a deformation

The category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{Sy}\,A}$ fits into a span

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{gr}\,A} \xleftarrow{\sigma\mapsto 0} \mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{Sy}\,A} \xrightarrow{\sigma\mapsto 1} \mathrm{L}\mathcal{M}\mathrm{od}_{A}\ ,$$

exhibiting $\mathrm{Sy}\,A$-modules as a *deformation* with generic fiber $\mathrm{L}\mathcal{M}\mathrm{od}_A$ and special fiber $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{gr}\,A}$.

Remark

If $\mathrm{L}\mathcal{M}\mathrm{od}_A^{\mathrm{Filt}}$ = filtered A-modules, span is equivalent to

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{gr}\,A} \xleftarrow{\mathrm{gr}} \mathrm{L}\mathcal{M}\mathrm{od}_A^{\mathrm{Filt}} \xrightarrow{\mathrm{colim}} \mathrm{L}\mathcal{M}\mathrm{od}_{A}\ .$$

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The $\mathbb{C}$-motivic Steenrod algebra

The classic Steenrod algebra

$\mathcal{A}$ is the $\mathbb{F}_2$-algebra given by:

1. Generators: Sq^r for $r \geq 0$, with $|\mathrm{Sq}^r| = -r$ and $\mathrm{Sq}^0 = 1$;
2. Relations: $\mathrm{Sq}^{2s-r-1}\mathrm{Sq}^s = \sum_i \binom{r-i-1}{i}\mathrm{Sq}^{2s-1-i}\mathrm{Sq}^{s-r+i}$ for $r \geq 0$.

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$\mathcal{A}_{\mathbb{C}}$ is the $\mathbb{F}_2[\tau]$-algebra, with $|\tau| = (0,-1)$, given by:

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The $\mathbb{C}$-motivic Steenrod algebra as a deformation

Last slide

$\mathcal{A}_{\mathbb{C}}$ is the same as $\mathcal{A}$, except generators have an extra bidegree, and have a τ to make relations homogeneous.

A filtration

Define the *weight filtration* on $\mathcal{A}$ by

$$\mathcal{A}_{\leq n} = \{\mathrm{Sq}^I : I = (r_1, \ldots, r_n) \text{ with } \Sigma_i \lfloor \tfrac{r_i}{2} \rfloor \leq n\} \subset \mathcal{A}.$$

Then with respect to this filtration we have

$$\mathrm{Sy}\,\mathcal{A} = \mathcal{A}_{\mathbb{C}} \qquad (\sigma \leftrightarrow \tau).$$

Side questions

1. Is there a more conceptual explanation for this filtration? (Idea: want it to be related to the Frobenius on $\mathcal{A}_*$).
2. Similarly $\mathcal{A}_{\mathbb{R}} = \mathrm{Sy}(\mathcal{A}_{\mathbb{R}}/(\rho = 1))$; is $\mathcal{A}_{\mathbb{R}}/(\rho = 1)$ recognizable?

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SS for the cohomology of a filtered algebra

Fix A = augmented k-algebra (maybe add finiteness), k a field.

Definition: cohomology algebra

The cohomology algebra $H^*(A)$ can be defined as:

$$H^*(A) = \mathrm{Ext}_A^*(k,k) = H_* C_A, \qquad C_A^n = \mathcal{M}\mathrm{od}_k(I(A)^{\otimes_k n}, k).$$

Now say $A = \mathrm{colim}_n A_{\leq n}$ filtered by subalgebras.

The filtration spectral sequence

Filtration on A gives filtration $C_A = \lim_m C_A[\leq m]$ by

$$C_A^n[\leq m] = \ker\left(C_A^n[\leq m] \to \mathcal{M}\mathrm{od}_k\Big(\sum_{r_1+\cdots+r_n\leq m} A_{\leq r_1}\otimes\cdots\otimes A_{\leq r_m}, k\Big)\right).$$

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$${}^{\mathrm{Filt}}E_1 = H^*(\mathrm{gr}\, A) \Rightarrow H^*(A).$$

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The Bockstein spectral sequence

Have a filtration

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fibers are

$$\mathcal{E}\mathrm{xt}_{\mathrm{Sy}\,A}(k[\sigma], k\{\sigma^n\}) \simeq \mathcal{E}\mathrm{xt}_{\mathrm{Sy}\,A/(\sigma)}(k, k).$$

This gives the *σ-Bockstein spectral sequence*

$${}^{\mathrm{BSS}}E_1 = H^*(\mathrm{Sy}\,A/(\sigma))[\sigma] \Rightarrow H^*(\mathrm{Sy}\,A).$$

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This gives the σ-*Bockstein spectral sequence*

$${}^{\mathrm{BSS}}E_1 = H^*(\mathrm{Sy}\, A/(\sigma))[\sigma] \Rightarrow H^*(\mathrm{Sy}\, A).$$

But filtered algebras and deformations are the same

Last two slides

Have the filtration SS and Bockstein SS:

$$ {}^{\mathrm{Filt}}E_1 = H^*(\mathrm{gr}\, A) \Rightarrow H^*(A); $$
$$ {}^{\mathrm{BSS}}E_1 = H^*(\mathrm{Sy}\, A/(\sigma))[\sigma] \Rightarrow H^*(\mathrm{Sy}\, A). $$

Remember $\mathrm{Sy}\, A/(\sigma) = \mathrm{gr}\, A$.

Theorem

There is an isomorphism of spectral sequences

$$ {}^{\mathrm{BSS}}E_r[\sigma^{-1}] \cong {}^{\mathrm{Filt}}E_r \otimes \mathbb{Z}[\sigma^{\pm 1}]. $$

Upshot

As ${}^{\mathrm{BSS}}E_1$ is free over $k[\sigma]$, inverting σ loses no information. Thus the BSS and Filtration SS *carry the same information.*

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Cohomology of a deformation

Interpretation of theorem

1. As we're over a field, $H^*(\mathrm{Sy}\,A) = {}^{\mathrm{BSS}}E_\infty$ as $k[\sigma]$-modules;
2. By construction, ${}^{\mathrm{BSS}}d_r$-differentials are those $d^r(x) = \sigma^r y$;
3. By ${}^{\mathrm{BSS}}d_r \leftrightarrow {}^{\mathrm{Filt}}d_r$, find

$$H^*(\mathrm{Sy}\,A) = ({}^{\mathrm{Filt}}Z_\infty \otimes \mathbb{Z}[\sigma])/(\sigma^r \cdot {}^{\mathrm{Filt}}d^r(x) = 0 : x \in {}^{\mathrm{Filt}}Z_r).$$

So additively $H^*(\mathrm{Sy}\,A)$ may be read off the filtration spectral sequence.

When not over a field, just have more extension problems.

Cohomology as a deformation

1. Have $H^*(\mathrm{Sy}\,A)[\sigma^{-1}] = H^*(A) \otimes \mathbb{Z}[\sigma^{\pm 1}]$;
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Where the magic happens

Last slide

Can read additive structure of $H^*(\mathrm{Sy}\, A)$ off SS $H^*(\mathrm{gr}\, A) \Rightarrow H^*(A)$.

Key observation

The multiplicative (and higher) structure of $H^*(\mathrm{Sy}\, A)$ tracks higher structure of the filtration SS, such as extension problems.

Examples

1. A relation $x \cdot y = \sigma \cdot z$ in $H^*(\mathrm{Sy}\, A)$ gives:
   1. A relation $x \cdot y = 0$ in ${}^{\mathrm{Filt}}E_\infty$;
   2. A relation $x \cdot y = z$ in $H^*(A)$;

   so describes a *hidden multiplicative extension* in the filtration SS.
2. Mixed relations (e.g. $x \cdot y = w + \sigma \cdot z$), Massey products, etc. track more complicated distinctions between ${}^{\mathrm{Filt}}E_\infty$ and $H^*(A)$.

$H^*(\mathrm{Sy}\, A)$ encodes the *full process* of computing with the filtration SS.

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Application to the Steenrod algebra

Question

If $H^*(\mathrm{Sy}\, A)$ just describes the filtration SS, why bother with it?

Recall $\mathcal{A}$ = Steenrod algebra, $\mathcal{A}_{\mathbb{C}} = \mathrm{Sy}\, \mathcal{A}$ for a suitable filtration.

The first Hopf element

1. Have $h_1 \in H^1(\mathcal{A})$. Have $h_1^4 = 0$, so $H^*(\mathcal{A})[h_1^{-1}] = 0$.
2. Lifts to $h_1 \in H^1(\mathcal{A}_{\mathbb{C}})$; now $h_1^4 \neq 0$, instead just $\tau \cdot h_1^4 = 0$.

Theorem (Guillou-Isaksen 2014)

$$H^*(\mathcal{A}_{\mathbb{C}})[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4, v_n : n \geq 2].$$

Corollary

Can detect various relations in $H^*(\mathcal{A})$ using the zigzag

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On beyond ordinary algebra

Observation

1. Went from filtration on A to filtration on C_A (or $\mathcal{E}\mathrm{xt}_A(k,k)$);
2. It's the homotopical filtration that did all the work.

So we should work with *filtered spectra*.

Filtered spectra: quick facts

1. (Lurie) Symmetric monoidal equivalence $\mathrm{Sy}\colon \mathcal{S}\mathrm{p}_{\mathrm{Filt}} \simeq \mathcal{M}\mathrm{od}_{\mathbb{S}[\sigma]}(\mathcal{S}\mathrm{p}_{\mathrm{gr}})$
2. $C(\sigma^n) = \mathbb{S}[\sigma]/(\sigma^n)$ is $\mathbb{E}_\infty$, with $C(\sigma) = $ unit of $\mathcal{S}\mathrm{p}_{\mathrm{gr}}$;
3. Given $X = \mathrm{colim}_n X(n)$ filtered spectrum, have

$$(X/(\sigma^n))(p) = (X \otimes C(\sigma^n))(p) = \mathrm{Cof}(X(p-n) \to X(p)).$$

This, and what follows, generalizes to other stable categories.

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Review: spectral sequences

Notation for $X \in \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$

Write $X(p,q) = \mathrm{Cof}(X(p) \to X(q))$, with $X(\infty) = \mathrm{colim}\, X$.

Theorem

There is a spectral sequence

$$E^1_{p,q} = \pi_q X(p-1,p) \Rightarrow \pi_q X(\infty), \qquad d_r^{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1};$$

Explanation, or, what is a spectral sequence?

1. Each (E^r, d^r) is a chain complex, i.e. $d^r \circ d^r = 0$;
2. Isomorphisms $E^{r+1} = H_*(E^r, d^r) = Z^r/B^r$ (but d^{r+1} is extra).

So each E^{r+n} is a *subquotient* of E^r. In good cases (i.p. $X(-\infty) = 0$):

3. With $E^\infty = (\cap_r Z^r)/(\cup_r B^r)$ and $F^p = \mathrm{Im}(\pi_q X(p) \to \pi_q X(\infty))$,

$$E^\infty = \mathrm{gr}\, \pi_* X(\infty).$$

All spectral sequences will be considered convergent.

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Review: construction of the spectral sequence

Construction

For $p \leq q \leq r$, have cofibering

$$X(p,q) \to X(p,r) \to X(q,r).$$

If we define

$$E^r_{p,q} = \mathrm{Im}(\pi_q X(p-r,p) \to \pi_q X(p-1,p+r-1)),$$

we get $d^r\colon E^r_{p,q} \to E^r_{p-r,q-1}$ induced by a boundary map.

Example

Have $E^1_{p,q} = \pi_q X(p-1,p)$, and d^1 induced by boundary in X.

Terminology

The collection of $\pi_* X(p,q)$ and maps is a *Cartan-Eilenberg system.*
Can show $E^{r+1} = H_*(E^r, d^r)$, so get a spectral sequence.

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Review: detection

Definitions: filtration and detection

$x \in \pi_q X(\infty)$ is in *filtration* p if there is a lift

$$
\begin{array}{ccccccc}
 & & & & & & \mathbb{S}^q \\
 & & & & \overset{\widetilde{x}}{\swarrow} & & \downarrow x \\
\cdots \longrightarrow X(p-r) \longrightarrow \cdots & \longrightarrow & X(p) & \longrightarrow & \cdots & \longrightarrow & X(\infty) \\
 & & \downarrow & & & & \\
 & & X(p-r,p) & & & & \\
 & & \downarrow & & & & \\
 & & X(p-1,p+r-1) & & & &
\end{array}
.
$$

If x projects to $\overline{x} \in E^r_{p,q} \subset \pi_q X(p-1,p+r-1)$ nonzero, one says that x is *detected* by $\overline{x}$.

Review: hidden extensions

"Definition"

A *hidden additive extension* refers to situations like:

1. $x \in \pi_q X(\infty)$ with $2x \neq 0$;
2. x is detected by $\overline{x} \in E^{\infty}_{p,q}$, and $2\overline{x} = 0$.

In general hidden extensions are the failure of "$E_\infty = \pi_* X(\infty)$".

Closer look

Fix x as above. Then:

1. As x is in filtration p, lift to $\widetilde{x} \colon \mathbb{S}^q \to X(p)$;
2. As $2x \neq 0$, have $2\widetilde{x} \neq 0$;
3. As $2\overline{x} = 0$ in E^∞, have lift $2\widetilde{x} \colon \mathbb{S}^q \to X(p-1)$;

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Homotopy groups of a filtered spectrum

Definition

The bigraded homotopy groups of $X \in \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$ are $\pi_{s,c}X = \pi_s X(c)$. Thus $\pi_{s,*}X$ is a $\mathbb{Z}[\sigma]$-module, $|\sigma| = (0,1)$.

Example

Filtration on A gives filtration on C_A with $\pi_{*,*}C_A = H^*(\mathrm{Sy}\,A)$.

As a deformation

1. Have $\pi_{*,*}X[\sigma^{-1}] = \pi_* X(\infty) \otimes \mathbb{Z}[\sigma^{\pm 1}]$;
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The hidden extension revisited

1. $x \in \pi_q X(\infty)$ in filtration p: lifts to $\widetilde{x} \in \pi_{q,p}X$;
2. If $2\overline{x} = 0$ in E^∞: have σ-divisibility $2\widetilde{x} = \sigma \cdot \widetilde{2x}$.

In general $\pi_{*,*}X$ records the computation of the spectral sequence $\pi_* \operatorname{gr} X \Rightarrow \pi_* X(\infty)$, e.g. σ^r-torsion corresponds to d_r-differentials.

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Homotopy groups of a filtered spectrum

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The bigraded homotopy groups of $X \in \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$ are $\pi_{s,c}X = \pi_s X(c)$. Thus $\pi_{s,*}X$ is a $\mathbb{Z}[\sigma]$-module, $|\sigma| = (0,1)$.

Example

Filtration on A gives filtration on C_A with $\pi_{*,*}C_A = H^*(\mathrm{Sy}\, A)$.

As a deformation

1. Have $\pi_{*,*}X[\sigma^{-1}] = \pi_* X(\infty) \otimes \mathbb{Z}[\sigma^{\pm 1}]$;
2. Have edge map $(\pi_{*,*}X)/(\sigma) \to E_\infty^{*,*}$.

The hidden extension revisited

1. $x \in \pi_q X(\infty)$ in filtration p: lifts to $\widetilde{x} \in \pi_{q,p}X$;
2. If $2\overline{x} = 0$ in E^∞: have σ-divisibility $2\widetilde{x} = \sigma \cdot \widetilde{2x}$.

In general $\pi_{*,*}X$ records the computation of the spectral sequence $\pi_* \operatorname{gr} X \Rightarrow \pi_* X(\infty)$, e.g. σ^r-torsion corresponds to d_r-differentials.

The trigraded spectral sequence

Filtrations

1. $X(\infty) = \operatorname{colim}_p X(p)$ gave spectral sequence for $\pi_* X(\infty)$;
2. $X(n) = \operatorname{colim}_{p \leq n} X(p)$ gives spectral sequence for $\pi_* X(n)$.

Description

The SS for $\pi_* X(n)$ looks like that for $\pi_* X(\infty)$, cut off at filtration n:

1. Stuff in filtration $> n$ is no longer present;
2. If $d_r(x)$ in filtration $> n$ for $X(\infty)$, then x is p.c. for $X(n)$.

Theorem (putting it all together)

Trigraded spectral sequence $E_1^{s,c,f} = (\pi_{s,c-f}(X/\sigma))\{\sigma^f\} \Rightarrow \pi_{s,c}X$.

Proof

This is just the σ-Bockstein spectral sequence

$$E_1^{s,c,*} = \pi_{s,c}X/(\sigma)[\sigma] \Rightarrow \pi_{s,c}X.$$

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Deformations of stable homotopy theories

Filtered ring spectra as deformations

For $R \in \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$ a filtered $\mathbb{A}_\infty$ ring, have

1. $R(\infty)$ an ordinary $\mathbb{A}_\infty$ ring spectrum;
2. $\operatorname{gr} R = R \otimes C(\sigma)$ a graded $\mathbb{A}_\infty$ ring spectrum.

Get *deformation* with generic fiber $R(\infty)$ and special fiber $\operatorname{gr} R$.

Filtered modules as a deformation

Have filtered module category $\mathrm{L}\mathcal{M}\mathrm{od}_R$, with span

$$\mathrm{L}\mathcal{M}\mathrm{od}_{R\otimes C(\sigma)} \xleftarrow{\sigma\mapsto 0} \mathrm{L}\mathcal{M}\mathrm{od}_R \xrightarrow{\sigma\mapsto 1} \mathrm{L}\mathcal{M}\mathrm{od}_{R(\infty)} :$$

A *deformation*, generic fiber $\mathrm{L}\mathcal{M}\mathrm{od}_{R(\infty)}$ and special fiber $\mathrm{L}\mathcal{M}\mathrm{od}_{R\otimes C(\sigma)}$. Deformation theory governed by "higher structure" of filtration SS

$$\pi_q \operatorname{gr} R \Rightarrow \pi_q R(\infty).$$

Can generalize to other settings.

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The trivial deformation

Observation

$\mathcal{S}\mathrm{p}_{\mathrm{Filt}} = \mathcal{M}\mathrm{od}_{\mathbb{S}[\sigma]}$ itelf has generic fiber $\mathcal{S}\mathrm{p}$ and special fiber $\mathcal{S}\mathrm{p}_{\mathrm{gr}}$.

The generalized Mahowald square

For $X \in \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$, if each $H_*X(p-1) \subset H_*X(p)$ (say $H = H\mathbb{F}_2$), get

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}(H_{*,*}\,\mathrm{gr}\,X)[\sigma] & \xrightarrow[\text{Bockstein SS}]{\text{Algebraic}} & \mathrm{Ext}_{\mathcal{A}}(H_{*,*}X) \\ \Downarrow \text{ASS} & & \Downarrow \text{ASS} \\ \pi_{*,*}\,\mathrm{gr}\,X[\sigma] & \xrightarrow{\text{Bockstein SS}} & \pi_{*,*}X \end{array}.$$

The diagonal: the filtration-complete Adams SS

Where $H_{\mathrm{Filt}} := H \otimes C(\sigma)$, have $\mathcal{A}_{\mathrm{Filt}} = \mathcal{A}[\epsilon]/(\epsilon^2)$ and

$$E_2 = \mathrm{Ext}_{\mathcal{A}_{\mathrm{Filt}}}(H_{*,*}\,\mathrm{gr}\,X) \Rightarrow \pi_{*,*}X.$$

When the Mahowald square is defined, have $E_2 = \mathrm{Ext}_{\mathcal{A}}(H_*\,\mathrm{gr}\,X)[\sigma]$.

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The t-structure deformation

Whitehead towers

There is a lax symmetric monoidal functor

$$W\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{\mathrm{Filt}}, \qquad W(X)_n = \tau_{\geq n} X.$$

Definition: synthetic R-modules

Given $\mathbb{A}_\infty$ ring R, get $\mathrm{Syn}_R = \mathrm{L}\mathcal{M}\mathrm{od}_{W(R)} =$ *synthetic R-modules.*
Deformation: generic fiber $\mathrm{L}\mathcal{M}\mathrm{od}_R$ and special fiber $\mathrm{L}\mathcal{M}\mathrm{od}_{R_*} = \mathcal{D}(R_*)$.

Example application

Have functor $W\colon \mathrm{L}\mathcal{M}\mathrm{od}_R \to \mathrm{Syn}_R$. For R commutative:

1. $(W(M) \otimes_{W(R)} W(N))[\sigma^{-1}] \simeq M \otimes_R N \in \mathcal{M}\mathrm{od}_R$;
2. gr strong sym. monoidal, so $\mathrm{gr}(W(M) \otimes_{W(R)} W(N)) \simeq M_* \otimes^{\mathbb{L}}_{R_*} N_*$.

Thus filtration SS = *Künneth spectral sequence.*

Remark: a more abstract construction

Can show $\mathrm{Syn}_R = \mathcal{S}\mathrm{p}$-valued modules of algebraic theory $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}$.

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Filtered model of Borel C_2-equivariant homotopy

Work with everything 2-complete.

Definition: the root sphere

Let $\mathbb{S}_{\mathrm{Root}}$ = filtered spectrum with $\mathbb{S}_{\mathrm{Root}}(n) = F(P^\infty_{-n}, \mathbb{S})$; write $\rho = \sigma$.

Facts

1. $\mathbb{S}_{\mathrm{Root}}$ is a filtered $\mathbb{E}_\infty$ ring spectrum (?), so get $\mathcal{S}\mathrm{p}_{\mathrm{Root}} := \mathcal{M}\mathrm{od}_{\mathbb{S}_{\mathrm{Root}}}$;
2. Have $\mathbb{S}_{\mathrm{Root}}/(\rho) = \mathbb{S}[\tau^{\pm 1}]$, and Lin's theorem: $\mathbb{S}_{\mathrm{Root}}[\rho^{-1}] \simeq \mathbb{S}[\rho^{\pm 1}]$;
3. Rmk: C_2 Segal conj. (Lin): $\pi_{s+c\sigma}\mathbb{S}_{C_2} = \pi_{s,c}\mathbb{S}_{\mathrm{Root}}$.

Reinterpretation

There is an equivalence of categories

$$\nu\colon \mathrm{Fun}(BC_2, \mathcal{S}\mathrm{p}) \simeq \mathcal{S}\mathrm{p}_{\mathrm{Root}}^{\mathrm{Cpl}(\rho)}, \qquad \nu(X)(n) = F(S^{-n\sigma}, X)^{\mathrm{h}C_2}.$$

Question

What does this look like for other groups?

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Borel C_2-equivariant homotopy as a deformation

Root invariants

Have filt SS = Atiyah-Hirezbruch SS $\pi_* \operatorname{gr} \mathbb{S}_{\mathrm{Root}} = \pi_* \mathbb{S}[\tau^{\pm 1}] \Rightarrow \pi_* \mathbb{S}$.
The *root invariants* $R(\alpha)$ of $\alpha \in \pi_* \mathbb{S}$ are those $\beta \in \pi_* \mathbb{S}$ that detect α.

Interpretation

Borel C_2-spectra are a deformation with:

1. Generic fiber $\mathcal{S}\mathrm{p}$, realized by Tate constr. $\mathrm{Fun}(BC_2, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$;
2. Special fiber $\mathcal{S}\mathrm{p}$, realized by the forgetful map $\mathrm{Fun}(BC_2, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$;
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Rmk: the Borel Steenrod algebra

Have $\mathcal{A}_{\mathrm{Root}} = \mathrm{Sy}(H_c^*(P_{-\infty}^{\infty}) \otimes' \mathcal{A})$ with $\mathcal{H}(\mathcal{A}_{\mathrm{Root}}) = \mathrm{Sy}_{\mathrm{Root}}(\mathcal{H}(\mathcal{A}))$, and

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Outside $\mathcal{S}\mathrm{p}_{\mathrm{Root}}$, get filt-complete ASS $H^*(\mathcal{A})[\tau^{\pm 1}, \rho] \Rightarrow \pi_{*,*} \mathbb{S}_{\mathrm{Root}}$.

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Décalage

Spectral sequence of a cosimplicial spectrum

Given $X\colon \Delta \to \mathcal{S}\mathrm{p}$, there is a spectral sequence

$$E_2 = H_* N(\pi_* X) \Rightarrow \pi_* \operatorname{Tot} X,$$

where N = chain complex of cosimplicial group $\pi_* X$.

Fact

There is lax symmetric monoidal $\operatorname{Dec}\colon \operatorname{Fun}(\Delta, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}_{\mathrm{Filt}}$ such that

1. $\operatorname{colim} \operatorname{Dec} X \approx \operatorname{Tot} X$;
2. Filtration SS E_1 = cosimplicial SS E_2.

Construction

Have $(\operatorname{Dec} X)(n) = \operatorname{Tot}(m \mapsto \tau_{\geq n} X(m))$.

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2. Filtration SS E_1 = cosimplicial SS E_2.

Construction

Have $(\operatorname{Dec} X)(n) = \operatorname{Tot}(m \mapsto \tau_{\geq n} X(m))$.

Synthetic spectra

Fix an $\mathbb{A}_\infty$ ring R.

The descent complex

Have an augmented cosimplicial $\mathbb{A}_\infty$ ring:

$$C_R = \ \mathbb{S} \longrightarrow R \rightrightarrows R \otimes R \Rrightarrow R \otimes R \otimes R \cdots$$

Totalization is $\mathbb{S}_R^\wedge = R$-nilpotent completion of $\mathbb{S}$.

Definition: synthetic spectra

Set $\mathrm{Sy}_R\,\mathbb{S} = \mathrm{Dec}(C_R)$; then R-synthetic spectra $= \mathcal{S}\mathrm{p}_R = \mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{Sy}_R\,\mathbb{S}}$.

R-based Adams-Novikov spectral sequence

Have $\mathrm{Sy}_R\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_R$ by $\mathrm{Sy}_R(X) = \mathrm{Dec}(X \otimes C_R)$. Filtration SS is $E_0^{p,q} = \pi_q(X \otimes R^{\otimes p+1}) \Rightarrow \pi_q X_R^\wedge$; this is the R-based ANSS.

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Synthetic spectra (cont.)

Question

$\mathrm{Sy}_R\,\mathbb{S}$ has generic fiber $\mathbb{S}^\wedge_R$, what is special fiber $\mathrm{Sy}_R\,\mathbb{S}\otimes C(\sigma)$?.

Flatness

If R_*R is flat over R_*, then

1. (R_*, R_*R) is a Hopf algebroid;
2. $\pi_*C_R =$ cobar complex for cohomology of (R_*, R_*R).

Thus $\mathrm{Sy}_R\,\mathbb{S}\otimes C(\sigma) =$ graded cohomology spectrum of (R_*, R_*R).

Consequence

For R_*R flat, $\mathcal{LM}\mathrm{od}_{\mathrm{Sy}_R\,\mathbb{S}\otimes C(\sigma)}$ is a htpy theory of (R_*, R_*R)-comodules.

Question

Exactly which homotopy theory is $\mathcal{LM}\mathrm{od}_{\mathrm{Sy}_R\,\mathbb{S}\otimes C(\sigma)}$?
(Is $\mathcal{LM}\mathrm{od}_{\mathrm{Sy}_R\,\mathbb{S}\otimes C(\sigma)} \simeq \mathcal{S}\mathrm{table}_{(R_*,R_*R)}$?)

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$\mathbb{F}_2$-synthetic spectra $\mathcal{S}\mathrm{p}_{\mathbb{F}_2}$

Example: 0-stem

1. Have $\mathbb{F}_2[h_0]$ = 0-stem of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$;
2. h_0 detects 2, i.e. hidden extension $2\colon 1 \to h_0$;
3. get $\pi_{0,*}\,\mathrm{Sy}_{\mathbb{F}_2}\,\mathbb{S} = \mathbb{Z}_2[\sigma, h_0]/(\sigma \cdot h_0 = 2)$; this is also $\pi_{*,*}\,\mathrm{Sy}_{\mathbb{F}_2}\,H\mathbb{Z}$.

Some computations

1. Burklund-Hahn-Senger (2019, 2020) compute Toda range $\pi_{\leq 19,*}\,\mathrm{Sy}_{\mathbb{F}_2}\,\mathbb{S}$, give applications to manifold geometry, ...;
2. Burklund (2020) uses $\mathrm{Sy}_{\mathbb{F}_2}\,\mathbb{S}$ to get last 2-extension in $\pi_{\leq 80}\mathbb{S}$;
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Also possible to define *unstable* $\mathbb{F}_p$-synthetic objects $\mathrm{Sy}^{\mathrm{un}}_{\mathbb{F}_p}\,X$.
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What does chromatic homotopy theory in $\mathcal{S}\mathrm{p}_{\mathbb{F}_p}$ look like?

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MU-synthetic spectra and $\mathbb{C}$-motivic homotopy

MU-synthetic category $\mathcal{S}\mathrm{p}_{MU}$ as a deformation

Generic fiber $\mathcal{S}\mathrm{p}$ and special fiber $\approx (MU_*, MU_*MU)$-comodules.

Even objects

Let $\mathcal{S}\mathrm{p}_{MU}^{\mathrm{even}} = X$ with $X(2n) = X(2n+1)$; then $\mathrm{Sy}_{MU}\,\mathbb{S} \in \mathcal{S}\mathrm{p}_{MU}^{\mathrm{even}}$.

Theorem (Gheorghe-Isaksen-Krause-Ricka 2018)

If we p-complete, there is an equivalence of categories $\mathrm{Sy}_{MU}^{\mathrm{even}} \simeq \mathcal{S}\mathrm{p}_{\mathbb{C}}^{\mathrm{cell}}$.

Application (Gheorghe-Isaksen-Krause-Ricka 2018)

$\mathrm{Sy}_{MU}(tmf)$ gives $\mathbb{C}$-motivic tmf with good computational properties.

Side question

Is there a good *unstable* MU-synthetic category?

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MU-synthetic spectra and the Adams SS

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Even if $\mathrm{Sy}_{MU}\,\mathbb{S}$ may be useful, why bother with $\mathcal{S}\mathrm{p}_{MU}$?

One answer

General tools for $\mathcal{S}\mathrm{p}$ upgrade to tools for $\mathcal{S}\mathrm{p}_{MU}$.

The MU-synthetic Steenrod algebra

Have $\mathrm{Sy}_{MU}\,H\mathbb{F}_2$, giving MU-synthetic Steenrod algebra $\mathcal{A}_{MU}$; in fact

$$\mathcal{A}_{MU} = \mathcal{A}_{\mathbb{C}}[\sigma]/(\sigma^2 = \tau).$$

This is a deformation of $\mathcal{A}$, i.e. $\mathcal{A}_{MU}[\sigma^{-1}] = \mathcal{A} \otimes \mathbb{Z}[\sigma^{\pm 1}]$.

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Get e.g. Adams SS $H^*(\mathcal{A}_{MU}) \Rightarrow \pi_{*,*}\,\mathrm{Sy}_{MU}\,\mathbb{S}$.

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Utility of the MU-synthetic Adams SS

The Miller square

Have a square of spectral sequences:

$$\begin{array}{ccc} H^*(\mathcal{A}_{MU}/(\sigma))[\sigma] & \xRightarrow{\sigma\text{-BSS}} & H^*(\mathcal{A}_{MU}) \\ \Downarrow{\scriptstyle \mathrm{Sy}_{MU}\text{-ASS}} & & \Downarrow{\scriptstyle \mathrm{Sy}_{MU}\text{-ASS}} \\ \pi_* \mathrm{Sy}_{MU}\, \mathbb{S}/(\sigma)[\sigma] & \xRightarrow{\sigma\text{-BSS}} & \pi_* \mathrm{Sy}_{MU}\, \mathbb{S} \end{array}.$$

1. By construction, bottom is the Adams-Novikov SS;
2. Right is a deformation returning classic ASS after inverting σ;
3. Left turns out to be algebraic Novikov SS;
4. Question: is top the CESS for frobenius extension $\mathcal{A}^\vee \to \mathcal{A}^\vee$?

Used (Isaksen-Wang-Xu 2020 with $\mathbb{S}_\mathbb{C}$) to compute $\pi_*\mathbb{S}$ to 90-stem.

Problem

Compute the 2-parameter deformation "$\pi_{*,*,*}\, \mathrm{Sy}_{\mathrm{Sy}_{MU}\,\mathbb{F}_2}(\mathrm{Sy}_{MU}\,\mathbb{S})$".
This fully combines Adams filtration and Adams-Novikov filtration.

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